

ORBIFOLD SLOPE RATIONAL-CONNECTEDNESS

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ABSTRACT. We define, for smooth projective orbifold pairs (X, D) notions of ‘slope Rational connectedness’, and of orbifold ‘slope Rational quotient’. These notions extend to this larger context the classical notions of rationally connected manifold and ‘rational quotient’ (sometimes called ‘MRC fibration’). Our notions and proofs work entirely in characteristic zero, and are based on the consideration of foliations with minimal positive slope with respect to some suitable movable class. The existence of covering or connecting families of ‘orbifold rational curves’ is indeed presently unknown in the orbifold context, in situations analogous to the classical case, when $D = 0$. By contrast, the notions we introduce here, are checkable in practice and can certainly be used to show general properties expected from the existence of connecting families of ‘orbifold rational curves’. The proofs given here in the orbifold context provide new proofs in the classical case where $D = 0$, since the classical proofs did not adapt to this broader context.

1. INTRODUCTION

Let X be any connected complex projective manifold. Its *rational quotient map* $\rho : X \dashrightarrow R$ splits X into its antithetical parts: rationally connected (the fibres), and non-uniruled (the base R). This splitting can alternatively be defined according to the slope-positivity of the cotangent bundle relatively to movable classes, without referring to rational curves. This will be the point of view adopted here.

Indeed, rational-connectedness is also characterised by the existence of a movable class α on X such that $\mu_\alpha(\mathcal{Q}) > 0$, for any quotient \mathcal{Q} of the tangent bundle T_X (cf. Proposition 1.3 below). Replacing T_X by the orbifold tangent bundle of a smooth orbifold pair (X, D) leads to define the notion of *slope Rationally Connected orbifolds*.

The main objective of the present text is indeed to introduce this notion by several definitions shown to be equivalent, and to construct the rational quotient splitting in the category of smooth orbifold pairs (X, D) . This permits to formulate and extend to arbitrary smooth orbifold pairs results previously restricted to either manifolds without orbifold structure, or to orbifold pairs with pseudo-effective canonical bundles.

One of the main problems of birational geometry is to ‘decompose’ functorially (quasi)-projective manifolds, by suitable fibrations, into parts having a ‘signed’ canonical bundle, the rational quotient map being the first step of such a decomposition. This decomposition is (conditionally in an orbifold version of the $C_{n,m}$ conjecture) achieved in [9]. The ‘parts’ in the decomposition are, however, not manifolds, but orbifold pairs with a ‘signed’ canonical bundle. Understanding the

structure of general (quasi)-projective manifolds thus requires the consideration of the larger category of orbifold pairs. They also appear in questions seemingly independent from these structural considerations such as the solution of the Shafarevich-Viehweg ‘hyperbolicity conjecture’ in [15].

As said above, our considerations do not refer to rational curves. A crucial advantage is that only characteristic zero arguments are used. Moreover, many basic properties of rationally connected manifolds can be obtained using negativity properties of their cotangent or tensor bundles, and not rational curves on them ([4], [16]).

Let us now state more precisely the main results of the text.

In order to provide a more accurate description of our results, we will recall first some of the key notions.

Let (X, D) be a ‘smooth projective orbifold pair’. In our previous text [15] (see also [8] and [9]), we introduced orbifold (co)tangent sheaves for general smooth orbifold pairs as above, by lifting them to any Kawamata cover $\pi : X' \rightarrow (X, D)$ adapted to (X, D) . We thus obtained dual locally-free sheaves $\pi^*(T(X, D)) \subset \pi^*(TX)$ and $\pi^*(\Omega^1(X, D)) \supset \pi^*(\Omega_X^1)$. Their determinants are the π -liftings of the usual anticanonical and canonical \mathbb{Q} -bundles $\pm(K_X + D)$.

Let now α be a movable class on X (see [13] for this notion, and the ones to follow, which were introduced there). The notions of slope, (semi-)stability, Harder-Narasimhan filtrations of torsion-free sheaves \mathcal{E} on X are defined as in the classical case of polarisations, with the same properties¹. In particular, the property $\mu_{\alpha, \min}(\mathcal{E}) > 0$ means that any quotient \mathcal{Q} of \mathcal{E} has a strictly positive α -slope.

Our first main result is:

Theorem 1.1. *Let (X, D) be a smooth orbifold pair. The following properties are equivalent:*

1. *For any dominant rational map $f : X \dashrightarrow Z$ with connected fibres, the orbifold base (Z, D_Z) of any ‘neat model’ of (f, D) has a canonical bundle $K_Z + D_Z$ which is not pseudo-effective, if $\dim(Z) > 0$.*
2. *For any ample line bundle A on X , and some m_A , one has: $h^0(X', \pi^*(\otimes^m(\Omega^1(X, D))) \otimes \pi^*(A)) = 0, \forall m \geq m_A$.*
- 2'. *For any ample A on X , and some m_A , one has: $h^0(X', \text{Sym}^m(\pi^*(\Omega^p(X, D))) \otimes \pi^*(A)) = 0, \forall m \geq m_A, \forall p > 0$.*
3. *One has: $\mu_{\pi^*(\alpha), \min}(\pi^*(TX, D)) > 0$, for some movable class α on X . Moreover, the class α can be chosen to be ‘movable-big’ (i.e. interior to the movable cone of X).*

¹The restriction theorem of Mehta-Ramanathan however fails in this context.

A smooth projective orbifold pair satisfying these properties will be said to be ‘slope-Rationally Connected’, or *sRC* for short.

Observe that the Harder-Narasimhan filtration of $\pi^*(T(X, D))$ and its slopes relative to $\pi^*(\alpha)$ are independent of the choice of the ramified cover $\pi : X' \rightarrow X$, because of the conceptually crucial² complement to our previous text [15]:

Theorem 1.2. *Let (X, D) be a smooth projective orbifold pair, and $\pi : X' \rightarrow X$ a cover adapted to (X, D) . Let α be a movable class on X , and $\pi^*(\alpha)$ be its (movable) inverse image on X' . Let $HN_{\pi^*(\alpha)}(\pi^*(T(X, D)))$ be the Harder-Narasimhan filtration of $\pi^*(T(X, D))$ relative to $\pi^*(\alpha)$.*

There exists a filtration denoted $HN_\alpha(T(X, D))$ of TX such that: $[HN_{\pi^(\alpha)}(\pi^*(T(X, D)))]^{\text{sat}} = \pi^*(HN_\alpha(T(X, D)))$, the saturation being taken in $\pi^*(TX)$.*

Moreover, the filtration $HN_\alpha(TX)$, as well as the $\pi^(\alpha)$ -slopes of $HN_{\pi^*(\alpha)}(\pi^*(T(X, D)))$, do not depend on the choice of the adapted cover π .*

Said otherwise: the induced slopes and distributions on $(X - \text{Supp}(D))$ do not depend on π . See Theorem 5.4 and its corollary 5.5 for details.

Theorem 1.1 is (except for the Property 1) the exact analogue of the classical case when $D = 0$, since we showed in [15] (as a direct consequence of the central results results [27], [25], [20], [3], [12]):

Proposition 1.3. *For a smooth projective connected complex manifold X , the following 4 properties are equivalent:*

1. *X is rationally connected (i.e: any two points can be connected by some rational curve)*
2. *For any ample line bundle A on X , there is an $m(A) > 0$ such that $h^0(X, \otimes^m(\Omega_X^1) \otimes A) = 0, \forall m \geq m(A)$.*
- 2'. *For any ample line bundle A on X , there is an $m'(A) > 0$ such that $h^0(X, \text{Sym}^m(\Omega_X^p) \otimes A) = 0, \forall m \geq m'(A), \forall p > 0$.*
3. *For any dominant rational map $f : X \dashrightarrow Z$, with Z smooth and $\dim(Z) > 0$, K_Z is not pseudo-effective.*
4. *$\mu_{\alpha, \min}(TX) > 0$ for some movable class α on X (see below for these notions).*

An important example of ‘slope Rationally connected’ smooth orbifold is the following³:

Theorem 1.4. *Let (X, D) be a smooth orbifold pair which is klt⁴, and Fano (ie: $-(K_X + D)$ is ample on X). Then (X, D) is slope rationally connected.*

²The expectation is indeed that the geometry of (X, D) is a well-defined concept, independent on the auxiliary constructions used to define it.

³Answering a question of B. Claudon.

⁴This means that all coefficients of the components of D are strictly less than 1.

The conclusion in general fails when (X, D) is not klt, as shown by the following example of [9], 6.17, p. 859: (\mathbb{P}^2, D) if D is the union of 2 distinct lines: this has a birational orbifold model which maps onto (\mathbb{P}^1, δ) , where δ consists of 2 distinct reduced points, an orbifold with trivial canonical bundle.

The ‘Slope Rationally Connected Quotient’ in the category of smooth orbifold pairs takes the following form, entirely similar to the classical ‘Rational quotient’ recalled above (see [6], [25]):

Theorem 1.5. *Let (X, D) be smooth, complex projective and connected. There exists (on some suitable birational model) an orbifold morphism which is a fibration $\rho : (X, D) \rightarrow (R, D_R)$ onto its (smooth) orbifold base (R, D_R) which has the following two properties:*

1. *Its smooth orbifold fibres (X_r, D_r) are sRC.*
2. *$K_R + D_R$ is pseudo-effective.*

This fibration is unique, up to orbifold birational equivalence.

A similar orbifold fibration (with orbifold fibres having $\kappa^+ = -\infty$, and orbifold base $\kappa \geq 0$) was defined in [8], [9] conditionally in either an orbifold version $C_{n,m}^{orb}$ of Iitaka’s $C_{n,m}$ conjecture, or in the ‘non-vanishing’ conjecture. The present text permits to give an unconditional (conjecturally equivalent) definition. See Section 13 for some brief details.

We showed in [15] that the orbifold cotangent bundle is ‘birationally stable, that is: $\nu^+(X, D) = \nu(X, D)$ when $K_X + D$ is pseudo-effective. We refer to §9 for the definitions. The Slope rationally connected quotient permits to describe the invariant $\nu^+(X, D)$ in the general case: $\nu^+(X, D) = \nu(R, D_R)$ (Theorem 9.8).

Our next statement shows that the notion of *slope Rationally Connected orbifold* permits to strengthen a former result of [15] (see Theorem 8.1 below for the proof, and more details):

Theorem 1.6. *Assume that $\mathcal{F}_D \subset \pi^*(T(X, D))$ is a D -foliation, and that $\mu_{\alpha', \min}(\mathcal{F}_D) > 0$ for some movable class α on X , where $\alpha' := \pi^*(\alpha)$. Then:*

1. *\mathcal{F} is algebraic, let $f : X \dashrightarrow Z$, be such that $\mathcal{F} = \text{Ker}(df)$.*
2. *On any ‘neat’ orbifold birational model $f' : (X', D') \rightarrow Z'$ of f , the generic orbifold fibre (X'_z, D'_z) of f' is sRC.*

Conversely, if (f, D) possesses the above property 2, the D -foliation \mathcal{F}_D associated to it⁵ has $\mu_{\alpha', \min}(\mathcal{F}_D) > 0$ for some α movable on X , and for any Kawamata cover π adapted to (X, D) .

In [15], Theorem 1.4, the conclusion was only that $K_{X'_z} + D'_z$ was not pseudo-effective for $z \in Z'$ generic. It was also shown when $D = 0$ that X'_z was rationally connected (Theorem 1.1). The absence of this

⁵By the construction recalled before the statement of Theorem 8.1.

notion in the orbifold case made it impossible to state (and of course to prove) anything when $D \neq 0$.

As already said, we make here no reference to orbifold rational curves. These are defined (as in [11]) in §11 to which we refer for more details and the conjectural characterisation of slope orbifold rational connectedness (resp. uniruledness) in terms of connecting (resp. covering) families of orbifold rational curves. Let us notice that even the answer to the much simpler question below is presently unknown:

Question: Let (X, D) be smooth projective with $K_X + D$ not pseudo-effective. Is X covered by rational curves C with $(K_X + D).C < 0$?

A very weak and partial solution of the conjectures is given in §12, assuming a positive answer to this question.

We give now a very brief description of the main implication $1 \Rightarrow 3$ of theorem 1.1: it requires several steps and intermediate constructions. The main ingredient in the proof is Theorem 1.6: if $\mathcal{F} \subset \pi^*(T(X, D))$ is a D -foliation of positive minimal slope for $\pi^*(\alpha)$, where α some movable class on X , then \mathcal{F} descends to an algebraic foliation i.e. there exists a map $f : X \dashrightarrow Z$ such that $\mathcal{F}_X = \text{Ker}(df)$ and $\alpha.(K_{X/Z} + D) < 0$ (on any ‘neat’ model of f). Moreover, we have $\dim(Z) < n := \dim(X)$.

Our proof works by induction on $\dim(X)$. The first step is to chose α such that $\dim(Z) = p < n$ is maximal. Then we change α in such a way that $f_*(\alpha) = 0$, and moreover α is ‘big’ on the ‘general’ fibre X_z of f . The minimality of the rank of \mathcal{F} then permits to show that \mathcal{F} is still the maximal destabilizing foliation associated to the new α , and is α -semi-stable (of positive slope). If $\mathcal{F} = \pi^*(T(X, D))$, we are done. Otherwise, we get from the induction hypothesis a movable class β on Z such that $\mu_{p^*(\beta), \min}(p^*(T(Z, D_Z))) > 0$, where $p : Z' \rightarrow (Z, D_Z)$ is a Kawamata cover adapted to (Z, D_Z) , the orbifold base of a ‘neat model’ $f : (X, D) \rightarrow Z$ of our initial f . We then chose a movable lifting β_X of β to X such that $f_*(\beta_X) = \beta$, and show that: $\mu_{\gamma, \min}(\pi^*(T(X, D))) > 0$, where $\gamma := \lambda.\alpha + \beta_X$ and $\lambda > 0$ is sufficiently big. It is here that the relative bigness of α is needed. Some additional technical difficulties arise from the fact that the image of $\pi^*(df) : \pi^*(T(X, D)) \rightarrow (f \circ \pi)^*(T(Z, D_Z))$ is not surjective on some f -vertical divisors. They are handled by using a ‘negativity-type’ lemma.

The structure of the text is described in the table of content.

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the statement an initial version of Theorem 6.7, and for the proof of Proposition 12.13, on which Theorem 12.12 relies.

The results of the present text deeply depend on former articles [15] and [13] written in collaboration with him and Thomas Peternell, respectively.

We now start by several technical sections needed in the proof of Theorem 1.1.

2. ORBIFOLD MORPHISMS

2.1. Orbifold morphisms. We first recall some notions introduced in [9], §2 and §3, which are needed here⁶.

Let (X, D) be an orbifold pair; this consists of a connected complex projective normal variety X , together with an ‘orbifold divisor’

$$(1) \quad D := \sum_{F \subset X} c_D(F) \cdot F,$$

where the components F are all the pairwise distinct irreducible (Weil) divisors on X , and the coefficients $c_D(F)$ are zero for all but a finite number of divisors F . The coefficients $c_D(F) = \frac{a}{b} \in [0, 1] \cap \mathbb{Q}$ are, moreover, rational numbers with a, b coprime positive integers.

We will need to also write: $c_D(F) = 1 - \frac{1}{m_D(F)}$, in terms of: $m_D(F) = (1 - c_D(F))^{-1}$, the ‘multiplicities of the F ’s in D ’, with the convention that $+\infty = \frac{1}{0}$ is in \mathbb{Q} . The correspondence between coefficients and multiplicities is bijective and strictly increasing. Any irreducible divisor F of X not in the support of D , has thus coefficient $c_D(F) := 0$, and so multiplicity $m_D(F) = 1$ in D .

The multiplicities have a natural geometric meaning: for example, if all $c_D(F)$ are of the form $(1 - \frac{1}{m_F})$, for integers m_F , then D is a ramification divisor. Moreover, the geometric functorial properties in the ‘orbifold category’ are expressed in terms of multiplicities, not in terms of coefficients (see below).

The pairs (X, D) will be usually be supposed to be ‘smooth’, meaning that X is smooth, and that the support $\text{Supp}(D) := \cup_{\{F \subset X : c_D(F) \neq 0\}} F$ has simple normal crossings. These ‘smooth’ orbifolds will now be turned, following [9], into a category.

2.2. Definitions. Let $f : X \rightarrow Z$ be a regular map *with connected fibres* between two complex connected normal projective varieties. Assume that orbifold structures (X, D_X) and (Z, D_Z) are given on X and Z , respectively. For any irreducible divisor E on Z , let:

⁶All definitions and properties stated here work for compact complex normal spaces, and even, more generally for proper holomorphic fibrations. But we shall need them only in the projective case.

$$(2) \quad f^*(E) := \sum_{F \subset X | f(F)=E} c_{f^*(E)}(F) \cdot F + R,$$

where $R \subset X$ is an f -exceptional effective divisor (ie: such that $f(R) \subsetneq E$ is of codimension at least 2 in Z). Notice that $R = 0$ if f is flat.

If E is \mathbb{Q} -Cartier on Z , in particular if Z is smooth, or when f has equidimensional fibres (for example, if it is flat), we can more precisely define the coefficients $c_{f^*(E)}(F)$ for any component F of $f^{-1}(E)$ of the scheme-theoretic f -inverse of F :

$$(3) \quad f^*(E) := \sum_{F \subset X} c_{f^*(E)}(F) \cdot F$$

In general, when $f : X \rightarrow Z$ is a fibration, and when the coefficient $t := c_{f^*(E)}(F)$ is defined for $f(F) \subset E$, we shall simply write: $f^*(E) = t.F + \dots$ to isolate the component F from the other components of $f^{-1}(E)$.

Definition 2.1. ([9], *Définition 2.3*) We say that f is an ‘orbifold morphism’ if, for any $E \subset Z$, we have:

$$(4) \quad c_{f^*(E)}(F) \cdot m_{D_X}(F) \geq m_{D_Z}(E),$$

for each F such that $c_{f^*(E)}(F) > 0$, provided $c_{f^*(E)}(F)$ is well-defined for each Weil divisor E in Z and each Weil divisor F in X . The two main situations considered here are either when Z is smooth, or when f has equidimensional fibres.

The orbifold morphism $f : (X, D_X) \rightarrow (Z, D_Z)$ is said to be an ‘orbifold birational equivalence’ if $f : X \rightarrow Z$ is birational, and if $D_Z = f_*(D_X)$. We have here to assume that Z is smooth (or \mathbb{Q} -factorial) in order for this to make sense if $X \neq Z$. The ‘orbifold birational equivalence relation’ is the one generated by this binary symmetric relation.

The orbifold morphism f is said to be ‘neat’ if (X, D_X) and (Z, D_Z) are smooth, and if, moreover, $f^{-1}[f(\text{Supp}(D)) \cup D_f]$ is of simple normal crossings, where D_f is the divisor on X where df , the derivative of f , is not of maximal rank.

The reason for these definitions will appear below, when composition of fibrations are considered. See [9] for further details.

Notice that, $\text{Supp}(D_Z) \subset f(\text{Supp}(D_X))$, and $D_Z \leq f_*(D_X)$ if f is birational, but even if $D_Z = f_*(D_X)$, it is not true that $f : (X, D_X) \rightarrow (Z, D_Z)$ is an orbifold morphism, in general.

Remark 2.2. Let us give an example (extracted from [9], and inspired by the Cremona transformation) which shows that if $(X, D_i), i = 1, 2$ are smooth and orbifold-birationally equivalent to some smooth

(X, D) for some orbifold divisors D_i, D on the same X , and if $D^+ = \sup(D_1, D_2)$, it may happen that (X, D^+) is no longer orbifold birational to them.

Let $b : X \rightarrow \mathbb{P}^2$ be the blow-up in 3 points a, b, c in general position, together with the 3 exceptional curves A, B, C . Let α, β, γ be the three lines going through two of these points, and α', β', γ' their inverse images in X . We thus get a new map $b' : X \rightarrow (\mathbb{P}^2)'$ contracting the 3 curves α', β', γ' . Consider the three (reduced) orbifold divisors $D := A + B + C$, $D' := \alpha' + \beta' + \gamma'$, and $D^+ := \sup\{D, D'\} = D + D'$ on X . Then $b : (X, D) \rightarrow \mathbb{P}^2$, $b' : (X, D') \rightarrow (\mathbb{P}^2)'$ are birational orbifold equivalences (the bases being equipped with the zero orbifold divisors). Since so are $b : X \rightarrow \mathbb{P}^2$, and $b' : X \rightarrow (\mathbb{P}^2)'$, the identity map of X induces orbifold birational equivalences: $(X, D) \cong X \cong (X, D')$ as well. Let $\Delta := b_*(D')$ and $\Delta' := (b')_*(D)$ be orbifold divisors on \mathbb{P}^2 and $(\mathbb{P}^2)'$ respectively. Then $b : (X, D^+) \rightarrow (\mathbb{P}^2, \Delta)$ and $b' : (X, D^+) \rightarrow ((\mathbb{P}^2)', \Delta')$ are also birational orbifold equivalences, but $b : (X, D') \rightarrow (\mathbb{P}^2, \Delta)$ and $b' : (X, D) \rightarrow ((\mathbb{P}^2)', \Delta')$ are not orbifold morphisms, although they map to their orbifold bases (defined below). Notice also that D^+ is **not** orbifold-birationally equivalent to $(X, D), (X, D'), X$, since: $\kappa(X, D^+) = 0 > -\infty = \kappa(X, D) = \kappa(X, D')$, and birational orbifold equivalence preserves the Kodaira dimension (and more generally, the differentials, see Proposition 2.11 below).

2.3. Orbifold base of a fibration. Let a fibration $f : X \rightarrow Z$ be given, X, Z being normal connected, and X moreover equipped with an orbifold structure (X, D_X) .

Definition 2.3. ([9], Définition 3.2) Let then $E \subset Z$ be an irreducible divisor, and, as defined above:

$$(5) \quad f^*(E) = R + \sum_{f(F)=E} c_{f^*(E)}(F) \cdot F,$$

where R is an f -exceptional divisor (i.e: $\text{codim}_X(f(R)) \geq 2$).

Define the multiplicity $m_{(f, D_X)}(E)$ of E relative to (f, D_X) by:

$$(6) \quad m_{(f, D_X)}(E) := \inf_{f(F)=E} c_{f^*(E)}(F) \cdot m_D(F),$$

and the ‘orbifold base’ $D_Z(f, D_X) = D_Z$ on Z by the equality:

$$(7) \quad D_Z := \sum_{E \subset Z} \left(1 - \frac{1}{m_{(f, D_X)}(E)}\right) \cdot E$$

The sum above being finite, this is an orbifold divisor on Z .

In general, $f : (X, D_X) \rightarrow (Z, D_Z)$ is not an orbifold morphism, because the multiplicities on the f -exceptional divisors of X are not taken into account, and may be too small. However, it is always possible to obtain

an orbifold morphism simply by flattening f and increasing sufficiently the multiplicities of D_X on the exceptional divisors of f .

Lemma 2.4. ([9], Proposition 3.10) *Let $f : (X, D_X) \rightarrow Z$ be a fibration, with (X, D_X) smooth, and Z normal. There exists a birational orbifold morphism $g : (X', D') \rightarrow (X, D_X)$ and a modification $h : Z' \rightarrow Z$, with Z' smooth, together with a fibration $f' : (X', D') \rightarrow Z'$ such that:*

- (1) $f \circ g = h \circ f'$.
- (2) *The orbifold base $(Z', D_{Z'})$ of $f' : (X', D') \rightarrow Z'$ is smooth.*
- (3) $f' : (X', D') \rightarrow (Z', D_{Z'})$ *is a neat orbifold morphism.*

2.4. Composition of fibrations. In this subsection we consider two fibrations $f : (X, D_X) \rightarrow Y$ and $g : Y \rightarrow Z$ with (X, D_X) , as well as Y and Z smooth. We aim at determining the orbifold base $(Z, D_{(g \circ f, D)})$ of the composition $(g \circ f, D_X)$ in the ‘smooth orbifold category’. The natural candidate is the orbifold base $(Z, D_{(g, D_Y)})$ of g , with (Y, D_Y) the orbifold base of (f, D_X) , that is: $D_Y := D_{(f, D_X)}$.

The two orbifold bases $(Z, D_{(g \circ f, D)})$ and (Y, D_Y) on Z are different: one always has: $D_{(g \circ f, D_X)} \leq D_{(g, D_Y)}$ with strict inequality in general. This is because of divisors $F \subset X$ of small multiplicity which are f -exceptional, but not $g \circ f$ -exceptional. This phenomenon is however excluded when f is an ‘orbifold’ morphism:

Proposition 2.5. ([9], Proposition 3.14). *Let $f : (X, D_X) \rightarrow Y$ and $g : Y \rightarrow Z$ be fibrations. Let (Y, D_Y) be the orbifold base of (f, D_X) . Assume that the induced map $f : (X, D_X) \rightarrow (Y, D_Y)$ is an orbifold morphism. Then we have $(Z, D_{(g \circ f, D)}) = (Z, D_{(g, D_Y)})$.*

2.5. Birational equivalence of fibrations. The example given in 2.2 shows that birational orbifold equivalence is not preserved by taking the smallest orbifold divisor dominating given ones. This drastically differs from the classical case of manifolds, and shows that some more care is needed here. Our objective here is the following result 2.6 and 2.10 (which answers the Question 3.12 raised in [9], and refines Corollaire 4.14 there).

Theorem 2.6. *Let $f_i : (X, D) \rightarrow (Z_i, D_i), i = 1, 2$ be fibrations, which are orbifold morphisms, with Z_1 and Z_2 smooth, where $D_i = D_{(f_i, D)}$ are the orbifold bases of $(f_i, D), i = 1, 2$. Assume that the fibrations f_i are birationally equivalent (ie: $Z'' := (f_1 \times f_2(X)) \subset (Z_1 \times Z_2)$ is the graph of a birational map between Z_1 and Z_2). Then (Z_1, D_1) and (Z_2, D_2) are birationally equivalent.*

Proof. Let Z' be the normalisation of Z'' defined above, together with the two natural projections $g_i : Z' \rightarrow Z_i$. The conclusion then follows from the following Lemma 2.7 applied to the g_i . \square

Lemma 2.7. *Let $f' : (X, D) \rightarrow Z'$ be a fibration between normal connected projective complex spaces, and $g : Z' \rightarrow Z$ be birational, with*

Z smooth. Assume that $f : (X, D) \rightarrow (Z, D_Z)$ is an orbifold morphism, where D_Z is the orbifold base of (f, D) . Let $D_{Z'}$ be the orbifold base of $f' : (X, D) \rightarrow Z'$. Then $g : (Z', D_{Z'}) \rightarrow (Z, D_Z)$ is a birational orbifold equivalence.

Proof. Let $E \subset Z$ and $E' \subset Z'$ be irreducible divisors with $g(E') \subset E$. We thus have: $g^*(E) = c.E' + \dots$ for some integer $c > 0$, since Z is smooth. We have to show that $c.m_{D_Z}(E') \geq m_{D_Z}(E)$, with equality if $g(E') = E$, in which case we have $c = 1$.

Let thus $F \subset X$ an irreducible divisor such that $g'(F) = E'$. We have: $f^*(E) = t.F' + \dots$, and $f'^*(E') = t'.F'$. Moreover, $t = c.t'$ since $f^*(E) = (f')^*(g^*(E))$, and since $f'(F) = E'$. On the other hand, since f is an orbifold morphism and $(Z', D_{Z'})$ is the orbifold base of (f', D) , we have: $t.m_D(F) \geq m_{D_Z}(E)$, and $t'.m_D(F) \geq m_{D_{Z'}}(E')$.

We distinguish the two cases: $g(E) = E'$ and $g(E') \neq E$.

Assume first⁷ that $g(E') = E$, so that $c = 1$ and $g : E' \rightarrow E$ is birational. We have $t = c.t' = t'$ in this case. We may then choose F in such a way that $t'.m_D(F) = m_{D_{Z'}}(E')$. Thus $m_{D_Z}(E) \leq m_{D_{Z'}}(E')$. By choosing now F such that $f(F) = E$ and $t.m_D(F) = m_{D_Z}(E)$, we get the reverse inequality, since $f'(F) = E'$, and $t'.m_D(F) \geq m_{D_{Z'}}(E')$.

If now $g(E') \neq E$, E' is g -exceptional. Let F be such that $t'.m_D(F) = m_{D_{Z'}}(E')$, and $t.m_D(F) = c.t'.m_D(F) = c.m_{D_{Z'}}(E')$. Since (f, D) is an orbifold morphism, and $f(F) \subset E$, we have: $t.m_D(F) \geq m_{D_Z}(E)$, and the inequality $m_{D_Z}(E) \leq c.m_{D_{Z'}}(E')$. \square

Lemma 2.8. *Let $(Z_i, D_i), i = 1, 2$ two orbifolds with Z_i smooth. Let $g_i : Z \rightarrow Z_i$ be birational, with Z normal and D be an orbifold divisor on Z . Assume that $g_i : (Z, D) \rightarrow (Z_i, D_i)$ are birational orbifold morphisms, with $(g_i)_*(D) = D_i$. Let $g : Z' \rightarrow Z$ be any desingularisation. There exists an orbifold divisor D' on D such that $g_*(D') = D$ and $g_i \circ g : (Z', D') \rightarrow (Z_i, D_i), i = 1, 2$ are birational orbifold equivalences.*

Proof. All g -exceptional divisors on Z' are $g_i \circ g$ -exceptional, for $i = 1, 2$. By putting on them sufficiently large multiplicities we can thus make $(g_i \circ g, D')$ orbifold morphisms. Because this does not change the orbifold bases, we thus still have $(g_i \circ g)_*(D') = D_i, i = 1, 2$, and these two maps are birational orbifold equivalences. \square

Lemma 2.9. *Let $f : (X, D) \rightarrow (Z, D_Z)$ be a fibration which is an orbifold morphism onto its orbifold base (Z, D_Z) , with Z smooth. Let $g : (Z', D_{Z'}) \rightarrow (Z, D_Z)$ be a birational orbifold equivalence with Z' smooth. There exists a birational orbifold equivalence $h : (X', D') \rightarrow (X, D)$ with X smooth such that $f \circ h = g \circ f'$ for some fibration $f' : (X', D') \rightarrow (Z', D_{Z'})$ which is an orbifold morphism onto its orbifold base.*

⁷In this case we do not use the fact that (f, D) is an orbifold morphism.

Proof. Take any birational $h : X' \rightarrow X$ with X' smooth such that $f' : X' \rightarrow Z'$ exists with $f \circ h = g \circ f'$. For each irreducible divisor $F' \subset X'$ which is h -exceptional, its image by f' is contained in or equal to some irreducible g -exceptional divisor $E' \subset Z'$. By putting on F' some sufficiently large multiplicity, we get a suitable orbifold structure D' on X' . (The details are easy, and one can make explicit the smallest such orbifold structure). \square

Theorem 2.10. *Let $f_i : (X_i, D_i) \rightarrow (Z, D_{Z_i}), i = 1, 2$ be two fibrations which are orbifold morphisms on their orbifold bases, with $Z_i, i = 1, 2$ smooth. Assume that $(X_i, D_i), i = 1, 2$ are birationally orbifold equivalent and that f_i are birationally equivalent (ie: so are X_i and f_i as seen on the graph of a birational equivalence of X_1 and X_2). Then $(Z_i, D_{Z_i}), i = 1, 2$ are birationally orbifold equivalent.*

Proof. By the definition of birational orbifold equivalence, we may assume the existence of some birational orbifold equivalence $h : (X_1, D_1) \rightarrow (X_2, D_2)$. The composition $f_2 \circ h : (X_1, D_1) \rightarrow (Z_2, D_{Z_2})$ is thus an orbifold morphism onto its orbifold base. The conclusion then follows from Theorem 2.6, the lemmas 2.8 and 2.9 showing that the birational orbifold equivalence between (Z_1, D_1) and Z_2, D_2 can be realised through an orbifold $(Z', D_{Z'})$ with Z' smooth. \square

2.6. Differentials and orbifold morphisms. We show here a crucial functoriality property of differentials for orbifold morphisms. This property is actually almost a characterisation of orbifold morphisms (see [9], Proposition 2.10, for a similar result).

Let $f : (X, D_X) \rightarrow (Z, D_Z)$ be an orbifold morphism, and $\pi : Y \rightarrow X$ and $p : W \rightarrow Z$ be Kawamata covers adapted to (X, D_X) and (Z, D_Z) respectively. Let G and H be the finite groups acting on Y and W respectively. Let

$$(8) \quad T := (Y \times_Z W)^n$$

be the normalisation of any component of the fibred product, together with the natural projections $\rho : T \rightarrow Y$ and $\tau : T \rightarrow W$. The projection $\pi \circ \tau : T \rightarrow X$ is still Galois, with group L , normal in $(G \times H)$, the stabilizer of T in $(Y \times_Z W)^n$, the group L being onto on both G and H . The components of $(Y \times_Z W)^n$ being exchanged under the operation of $(G \times H)/L$.

Let $df : TX \rightarrow f^*(TZ)$ be the derivative of f . Its lifting to T induces a map:

$$(9) \quad (\pi \circ \rho)^*(df) : (\pi \circ \rho)^*(TX) \rightarrow (f \circ \pi \circ \rho)^*(TZ) = (p \circ \tau)^*(TZ).$$

We recall (see [15]) that we have the natural inclusions $\pi^*(T(X, D)) \subset \pi^*(TX)$ as well as $p^*(T(Z, D_Z)) \subset p^*(TZ)$.

Proposition 2.11. *If f is an orbifold morphism, these maps extend to define natural maps of sheaves on T :*

$$(10) \quad (\pi \circ \rho)^*(df) : (\rho^*(\pi^*(T(X, D_X))) \rightarrow \tau^*(p^*(T(Z, D_Z))), \text{ and :}$$

$$(11) \quad (\pi \circ \rho)^*(df) : \tau^*(p^*(\Omega^1(Z, D_Z))) \rightarrow (\rho^*(\pi^*(\Omega^1(X, D_X))),$$

the latter one being injective.

Proof. Because both sheaves $\rho^*(\pi^*(T(X, D)))$ and $\tau^*(p^*(T(Z, D_Z)))$ are locally free, it is sufficient to establish the statement in codimension one on T , so in codimension one over Y , or X . It is thus sufficient to consider the situation over generic points of a divisor F of X which is contained in $f^{-1}(\text{Supp}(D_Z))$, since $\tau^*(p^*(T(Z, D_Z))) \subsetneq \tau^*(p^*(T(Z)))$ only there.

Let thus $x_0 \in F$ be such a point, and $z_0 = f(x_0) \in f(F) \subset E = f(F)$. We may thus assume that, in suitable coordinates $x = (x_1, \dots, x_n)$ for X near x_0 and $z = (z_1, \dots, z_p)$ for Z near $z_0 := f(x_0)$ the following hold true near x_0 and z_0 : $E = \{z_1 = z_2 = \dots = z_q = 0\}$, for some $1 \leq q \leq p$. Write $E_h = f^{-1}(z_h = 0)$, and: $f^*(E_h) = t_h \cdot F + \dots$, for some integer $t_h \geq 1$, for $h \in \{1, \dots, q\}$.

- (a) The divisor F is given by the equation $x_1 = 0$ near x_0 .
- (b) f is given, for holomorphic functions $g_\ell(x)$, $\ell \in \{1, \dots, p\}$, by:

$$f(x_1, \dots, x_n) = (x_1^{t_1} \cdot g_1, x_1^{t_2} \cdot g_2, \dots, x_1^{t_q} \cdot g_q, g_{q+1}, \dots, g_p)$$

Let m (resp. m'_h) be the multiplicity of F in D_X (resp. $f^{-1}(E_h)$, $\forall h \leq q$ in D_Z). Because f is an orbifold morphism, we also have:

- (c) $t_h \cdot m \geq m'_h$, thus: $(1 - \frac{1}{t_h \cdot m}) \geq (1 - \frac{1}{m'_h})$, $\forall h \leq q$.

By definition, the bundle $\pi^*(T(X, D))$ is ‘symbolically’ generated as an \mathcal{O}_Y -module by: $\pi^*(x_1^{(1-\frac{1}{m})} \cdot \partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})$ along F , and $p^*(T(Z, D_Z))$ is generated as an \mathcal{O}_W -module by $p^*(z_j^{(1-(1/m'_j))} \cdot \partial_{z_1}, \partial_{z_2}, \dots, \partial_{z_p})$ along E_1 , and by similar expressions involving the $p^*(z_j^{(1-(1/m'_j))} \cdot \partial_{z_j})$ along the other E'_j s.

The relations (b) imply that: $df(\partial_{x_1}) = (x_1^{s_j} \cdot g'_j \cdot f^*(\partial_{z_j}))_{j=1, \dots, p}$, where the $g'_j(x)$ are holomorphic, $s_j := t_h - 1$, for $j = 1, \dots, q$, and $s_j = 0$, for $j = q+1, \dots, p$.

$$\text{Thus: } df(x_1^{(1-\frac{1}{m})} \cdot \partial_{x_1}) = (x_1^{s_j+(1-\frac{1}{m})} \cdot g'_j \cdot f^*(\partial_{z_j}))_{j=1, \dots, p}.$$

Now observe that, for $j \leq q$, we have:

$$x_1^{t_j-1+(1-\frac{1}{m})} \cdot (f^*(z_j^{(1-(1/m'_j))})^{-1} = x_1^{t_j-1+(1-\frac{1}{m})-t_j \cdot (1-\frac{1}{m'_j})} = x_1^{(\frac{t_j}{m'_j}-\frac{1}{m})}.$$

The conclusion follows from (c) above, which implies that the ‘fractional vector field’ $df(x_1^{(1-\frac{1}{m})} \cdot \partial_{x_1})$ is in $f^*(T(Z, D_Z))$ along $E = f(F)$ if one considers its j -th component for $j \leq q$, and one uses that $F \subsetneq f^*(E_j)$ for $j = q+1, \dots, p$. The assertion for $df(\partial_{x_k})$, $k \geq 2$ follows

from the same computation of $df(\partial_{x_2})$, and simpler estimates of exponents, with s_j replaced by $s_j + 1$ for $j \leq q$, based on: $m'_j \geq m, \forall j \leq q$.

The dual case of the cotangent bundles is similar.

Now these ‘symbolic’ considerations easily imply the assertion, since the ramified covers π and p take the simplest possible form $x_1 = y_1^k, z_1 = w_1^\ell$, and $x_h = y_h, z_h = w_h$, for $h \geq 2$, near the points x_0, z_0 , for suitable integers k, ℓ . We leave the simple verifications to the reader. \square

The preceding computation moreover shows that⁸, since $\text{Supp}(D_X) \subset f^{-1}(D_Z)$ when (Z, D_Z) is the orbifold base of (f, D_X) , the quotient sheaf $\tau^*(p^*(T(Z, D_Z))/(\pi \circ \rho)^*(df)(\rho^*(\pi^*(T(X, D_X))))$ is supported on the inverse image in T of the components F of $f^{-1}(D_Z)$ for which: $t.m > m'$ in the above notations. The union $\tilde{F} \subset X$ is thus a divisor *partially supported on the fibres of f* in the following sense.

Definition 2.12. (See [8]) Let $f : X \rightarrow Z$ be as before. Let $F := \cup_{s \in S} F_s$ be a finite union of irreducible divisors of X . We say that ‘ F is partially supported on the fibres of f ’ if, for each $s \in S$, either F_s is f -exceptional (ie: $\text{codim}_Z(f(F_s)) \geq 2$), or if the following two properties hold:

1. $f(F_s) := E_s$ is a divisor of Z .
2. $f^{-1}(E_s) \cap F \subsetneq f^{-1}(E_s)$. (ie: at least one irreducible component of the RHS is not a component of F).

We thus have the following corollary.

Proposition 2.13. Let the situation $f : (X, D_X) \rightarrow (Z, D_Z)$ be as above, with (Z, D_Z) the orbifold base of (f, D_X) .

The quotient sheaf $\tau^*(p^*(T(Z, D_Z))/(\pi \circ \rho)^*(df)(\rho^*(\pi^*(T(X, D_X))))$ is supported in $(\pi \circ \rho)^{-1}(F)$, where $F \subset X$ is a divisor partially supported on the fibres of f .

The dual statement also holds similarly, the saturation being taken in $(\rho^*(\pi^*(\Omega^1(X, D_X))))$:

$$[(\pi \circ \rho)^*(df)(\tau^*(p^*(\Omega^1(Z, D_Z))))]^{sat} / ((\pi \circ \rho)^*(df)(\tau^*(p^*(\Omega^1(Z, D_Z))))$$
 is supported in $(\pi \circ \rho)^{-1}(F)$, where $F \subset X$ is a divisor partially supported on the fibres of f .

3. RELATIVE MOVABLE CLASSES

We consider here a fibration $f : X \rightarrow Z$ between two connected complex projective manifolds. We denote by $Mov(X) \subset N_1(X)$ the closed cone of movable classes on X , by $Mov^0(X)$ its interior, and similarly for Z . There is a natural surjective direct image map:

$$(12) \quad f_* : N_1(X) \rightarrow N_1(Z),$$

⁸Up to f -exceptional divisors $F \subset X$.

dual to the inverse image map $f^* : N^1(Z) \rightarrow N^1(X)$. The kernel $N_1(X/Z)$ of f_* is the orthogonal of the image of f^* .

Let $Mov(X/Z) := Mov(X) \cap N_1(X/Z)$ be the closed cone of relative movable classes for f , and let $N_1^0(X/Z) \subset N_1(X/Z)$ be the real vector space generated by $Mov(X/Z)$. We denote the interior of $Mov(X/Z)$ in $N_1^0(X/Z)$ by: $Mov^0(X/Z)$.

We state without proof a result not used in the sequel (the proof can be given by the argument proving Proposition 3.6.6 below):

Proposition 3.1. *The quotient vector space $N_1(X/Z)/N_1^0(X/Z)$ is generated by the classes of complete intersection curves on the irreducible divisors of X which are partially supported on the fibres of f . More precisely, if F is such an irreducible divisor, the corresponding classes are of the form⁹: $F.A^{n-r-1}.B^{r-1}$, if $f(F) := E, \dim(E) = r \geq 1$, and if A is ample on X , $B = f^*(B_E)$, B_E ample on E .*

For any smooth¹⁰ irreducible subvariety Y on X , we have natural inclusion maps $N_1(Y/Z) \rightarrow N_1(Y) \rightarrow N_1(X)$, where $N_1(Y/Z)$ denotes the Kernel of the composition maps $N_1(Y) \rightarrow N_1(X) \rightarrow N_1(Z)$. In general, of course, $Mov(Y/Z) := Mov(Y) \cap N_1(Y/Z)$ is not contained in $Mov(X/Z)$. This is, however, the case if, for example, $Y = X_y = f^{-1}(z), z \in Z$ is a ‘general’ fibre of f , in the following sense.

Definition 3.2. *A point $z \in Z$ of an irreducible complex space Z is ‘general’ if it lies in the complement of countably many specified strict Zariski closed subsets of Z , a fibre X_z of $f : X \rightarrow Z$ is ‘general’ if it lies over a ‘general’ point of $z \in Z$.*

Proposition 3.3. *Let $f : X \rightarrow Z$ be as above. For $z \in Z$ ‘general’, the natural inclusion map $j : X_z \rightarrow X$ induces an isomorphism of real vector spaces $j_* : N_1(X_z) \rightarrow N_1^0(X/Z)$, as well as bijections $j_* : Mov(X_z) \rightarrow Mov(X/Z)$, and $j_* : Mov^0(X_z) \rightarrow Mov^0(X/Z)$. For such a z , we can thus define a natural restriction map: $(j_*)^{-1} : Mov(X/Z) \rightarrow Mov(X_z)$. For each divisor F on X , and each $\alpha \in Mov(X/Z)$, we have further: $\alpha.F = \alpha_z.F_z$, for $\alpha_z := (j_*)^{-1}(\alpha)$, and $F_z := j^*(F)$.*

Proof. Let $C_1(X/Z)$ be the Chow-Barlet space of 1-dimensional algebraic cycles $\Gamma \subset X$ such that $f_*(\Gamma) = 0$, that is: which are contained in some fibre of f . The space $C_1(X/Z)$ has countably many irreducible components T_m , which are compact (ie: projective), and which parametrise an irreducible curve $C_t, t \in T_m$ generic. The support X_m of T_m , which is the union of all curves C_t parametrised by T_m , is thus a Zariski-closed subset of X . Let $Z_m := f(X_m), \forall m$. We now consider only the T'_m s such that $X_m = f^{-1}(Z_m)$, and define a point $z \in Z$ to be general if it does not belong to any of the Z_m such that

⁹ $F.A^{n-2}$ if $r = 0$.

¹⁰One can of course compose with a desingularisation of Y , also.

$Z_m = Z$. For such a point z , if $[C]$ is the class of an irreducible curve C contained in X_z , one thus has the equivalence between the two properties: C moves in a Z -covering algebraic family of curves, and: C moves in an X -covering family of curves such that $f_*([C]) = 0$.

This shows, for such a z , the existence and surjectivity of the map $j_* : \text{Mov}^0(X_z) \rightarrow \text{Mov}^0(X/Z)$, and thus also the surjectivity of $j_* : N_1(X_z) \rightarrow N_1(X/Z)$. In order to show the injectivity of this last map, we only need to show, by transposition, the surjectivity of the dual restriction map: $j^* : N^1(X/Z) := (N^1(X)/(f^*(N^1(Z)) + P)) \rightarrow N^1(X_z)$, where $P \subset N^1(X)$ is the vector space generated by irreducible divisors partially supported on fibres of f . This is achieved by the same argument as before, but applied to f -relative divisors on X . One however possibly needs in the process to suppress countably many new Zariski-closed subsets of Z .

For such a ‘general’ $z \in Z$, the map $j_* : N_1(X_z) \rightarrow N_1^0(X/Z)$ is thus linear bijective, hence homeomorphic. The induced map $j_* : \text{Mov}^0(X_z) \rightarrow \text{Mov}^0(X/Z)$ is thus bijective, and so is $j_* : \text{Mov}(X_z) \rightarrow \text{Mov}(X/Z)$. The last assertion is then obvious, since true for any curve contained in a fibre of X_z . \square

Definition 3.4. *A movable class on a smooth connected complex manifold X is said to be ‘big’ if it lies in the interior of the movable cone of X .*

Definition 3.5. *Let $f : X \rightarrow Z$ be a fibration as before. A class $\alpha \in \text{Mov}(X/Z)$ is said to be ‘big on the general fibre’ if its restriction $(j_*)^{-1}(\alpha) := \alpha_z$ is ‘big’ in X_z for $z \in Z$ ‘general’. We shall denote this simply by: α_z is ‘big’ ($z \in Z$ being implicitly assumed to be ‘general’).*

In this context, we have the following statement.

Proposition 3.6. *Let $f : X \rightarrow Z$ be as above, with $n := \dim(X)$, $d := n - \dim(Z)$. Let A, B be very ample divisors on X and Z , respectively, with $B_X := f^*(B)$. Let $\alpha \in \text{Mov}^0(X/Z)$, and $F := \cup_{s \in S} F_s$ be an effective reduced divisor on X , partially supported on the fibres of f .*

1. $A^{d-1}.B_X^{n-d} \in \text{Mov}^0(X/Z)$.
2. $N_1^0(X/Z)$ is generated as a real vector space by classes: $A^{d-1}.B^{n-d}$. If $\beta \in N_1(X/Z)$, we write $\alpha \geq \beta$ to mean that $\alpha - \beta \in \text{Mov}(X/Z)$.
3. There exists $\varepsilon > 0$ such that $\alpha \geq \varepsilon.A^{d-1}.B^{n-d}$.
4. For any $\vartheta \in N_1^0(X/Z)$, $k.\alpha \geq \vartheta, \forall k \geq k_0$, for some $k_0 \in \mathbb{R}$.
5. For any set of given reals $b_s, s \in S$, there exists $\vartheta \in N_1(X/Z)$ such that $\vartheta.F_s = b_s, \forall s \in S$.
6. Let $\beta \in \text{Mov}(Z)$. There exists $\beta' \in \text{Mov}(X)$ such that: $f_*(\beta') = \beta$, and moreover: $\beta'.F_s = 0, \forall s \in S$.

Proof. The assertion 1 is clear because complete intersections of ample divisor classes of X_z are big on X_z .

2 follows immediately from [21], Proposition 6.5, which states that the set of classes of the form $(A_z)^{d-1}$, with A_z an ample divisor on X_z is open in $N_1(X_z)$.

If α_z is big, $Mov^0(X_z)$ being open contains $\alpha_z - \varepsilon \cdot A^{d-1} \cdot B^{n-d}$ for some $\varepsilon > 0$. Hence 3, and 4, by the same argument applied to ϑ .

We now prove Claim 5, which is more involved. Write $B_X := f^*(B)$.

Let thus F_s be given. Define the class $\vartheta_s := F_s \cdot A^{d-1} \cdot B_X^{n-d-1} \in N_1(X/Z)$, by choosing a suitable representative of this class, one sees that it is a movable class of $N_1(F_s/E_s)$.

Moreover, it is easy to see also that we the orthogonality relations: $\vartheta_s \cdot F_{s'} = 0 = \vartheta_{s'} \cdot F_s$, unless $E_s = E_{s'}$.

The real numbers b_s being given, we will look for real numbers t_s such that $\vartheta := \sum_{s \in S} t_s \cdot \vartheta_s$, with the property that $\vartheta \cdot F_s = b_s, \forall s \in S$. By the orthogonality relations above, we just need to determine these real numbers in the case where $f(F_s) = E_s = E$ is the same E , for all $s \in S$. But in this case we can even reduce to: X is the surface X' cut out by (sufficiently generic and large multiple of) $A^{d-1} \cdot B_X^{n-d-1}$, and mapped by f to the curve $Z' := B^{n-d-1}$. In this case, because F is partially supported on the fibres of f , the same is true of $F' := F \cap X'$ over Z' . In this case, the assertion is a consequence of the ‘negativity lemma’ on quadratic forms of [2], Lemma 2.10, p. 19. This lemma indeed says that if $g : S \rightarrow B$ is a projective connected morphism from a smooth connected surface onto a curve with fibre $S_b := g^*(b), b \in B$, the intersection matrix of the components of S_b is semi-negative with Kernel S_b , which implies that it is strictly negative on the vector space generated by the components of any curve $C \subset S_b$ with support strictly contained in S_b .

We now prove assertion 6. Remark first that any $\beta \in Mov(Z)$ can be lifted to a $\beta'' \in Mov(X)$ such that $f_*(\beta'') = \beta$. This is clear if $\beta = [C_t]$ is the class of an algebraic irreducible family of curves $(C_t)_{t \in T}$ of Z which is Z -covering: one just need to choose, for example: $\beta'' := [A]^{d-1} \cdot [f^{-1}(C_t)]$, for $t \in T$ generic. This construction extends linearly (and thus continuously) to any $\beta \in Mov(Z)$.

Let us now consider an arbitrary irreducible divisor $E \subset Z$. Let $f^*(E) = \sum_{k \in K} c_k \cdot F_k + G'$, where G' is an effective f -exceptional divisor, while $f(F_k) = E$, and $c_k > 0, \forall k \in K$. Let us write: $K = J \cup L$, for some (possibly empty) $J \subset K$, and some (nonempty) $L \subset K$, where $J := (K \cap S)$ is the set of indices for the components of $f^*(E)$ which are contained in the divisor F which is ‘partially supported on the fibres of f ’.

Let, as above, $\vartheta_k := F_k \cdot A^{d-1} \cdot B_X^{n-d-1}$, for any $k \in K$. We obviously have: $f_*(\vartheta_k) = 0, \forall k \in K$. We shall chose $\beta' := \beta'' + \vartheta$, with $\vartheta := \sum_{k \in J} \nu_k \cdot \vartheta_k$. Because of the orthogonality relations mentioned above, $\beta' \cdot G = \beta'' \cdot G$ for any f -vertical divisor G not mapped onto E by f . In order to have β' movable, together with $\beta' \cdot F_s = 0, \forall s \in S$, we thus

only need to check the existence and non-negativity of the solutions $\nu_k, k \in J$ of the equations: $-(\sum_{k \in J} \nu_k \cdot \vartheta) \cdot F_j = \beta'' \cdot F_j \geq 0, \forall j \in J$. Indeed, the non-negativity of these coefficients ν_k then implies that $\beta' \cdot H \geq \beta'' \cdot H$ for any divisor H on X different from any of the F'_j s, which gives the movability of β' . By the negativity lemma above, we have the uniqueness and existence of the solutions $\nu_k, k \in J$. Now the non-negativity of all the ν'_k s simultaneously is precisely the assertion of the (elementary but crucial) [26], Corollary 4.2, pp. 112-113.

This solves the problem for the divisors F_s which are not f -exceptional. Let now F_s be an f -exceptional divisor, with $T := f(F_s)$ of codimension 2 at least in Z . Let $F' := \cup_{s' \in S' \subset S}$ be a connected component of the codimension 1 locus of $f^{-1}(T)$. Let $t := \dim(T) \geq 0$, and $B_T := f^*(B|_T)$. We then deal with F' exactly as before with $\cup_{k \in J} F_k$, just replacing the surface $X' := A^{d-1} \cdot B_X^{n-d-1}$ by $X_T := A^{n-t-1} \cdot B_T^{t-1}$ if $t \geq 1$, and by $X_T := A^{n-2}$ if $t = 0$, and the classes ϑ_s by the classes $\vartheta_{s'} := F_{s'} \cdot X_T$. The same negativity lemmas then apply. \square

4. POSITIVITY OF MINIMAL SLOPE

The following criteria will be crucial in the proof of Theorem 1.1 in the next section. We start with a basic situation which will be gradually extended in several small steps.

Let $\tau : T \rightarrow W'$ be a fibration (with connected fibres) between complex projective connected normal spaces. Let $r : R \rightarrow T$ and $s : W'' \rightarrow W'$ be resolutions of singularities.

The corresponding diagram is thus the following:

$$\begin{array}{ccccc} R & \xrightarrow{r} & T & \xrightarrow{\rho} & Y \\ & & \downarrow \tau & & \\ W'' & \xrightarrow{s} & W' & & \end{array}$$

Assume we have moreover movable classes α_R, β_R on R , and β'' on W'' such that: $(\tau \circ r)_*(\alpha_R) = 0$, and $(\tau \circ r)_*(\beta_R) = s_*(\beta'')$.

Let also an exact sequence of locally free sheaves be given on R :

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow (\tau \circ r)^*(\mathcal{G}) \rightarrow 0,$$

where \mathcal{G} is locally free on W' .

4.1. First Criterion.

Theorem 4.1. *In the preceding situation, we have: $\mu_{\gamma_R, \min}(\mathcal{E}) > 0$ if $\gamma_R := k \cdot \alpha_R + \beta_R$, for any real number $k > 0$ sufficiently large, and if moreover, the following properties are satisfied:*

1. $\mu_{\alpha_R, \min}(\mathcal{F}) > 0$, $\mu_{\beta'', \min}(s^*(\mathcal{G})) > 0$, and:
2. α_R is 'big' on the 'general' fibre $R_{w'}$ of $\tau \circ r$ (see definitions 3.4 and 3.2).

Proof. (of Theorem 4.1) Let \mathcal{Q} be a quotient of \mathcal{E} : it fits in an exact sequence:

$$0 \rightarrow \mathcal{Q}_F \rightarrow \mathcal{Q} \rightarrow \mathcal{Q}_G \rightarrow 0,$$

with \mathcal{Q}_F and \mathcal{Q}_G quotients of \mathcal{F} and \mathcal{G} respectively. Let $\gamma_R := k \cdot \alpha_R + \beta_R$ for some $k > 0$. Then $\mu_{\gamma_R}(\mathcal{Q})$ is a linear combination with positive coefficients of $\mu_{\gamma_R}(\mathcal{Q}_F)$ and $\mu_{\gamma_R}(\mathcal{Q}_G)$.

We shall treat them separately.

Lemma 4.2. *For $k > 0$ sufficiently large, $\mu_{\gamma_R}(\mathcal{Q}_F) = k \cdot \mu_{\alpha_R}(\mathcal{Q}_F) + \mu_{\beta_R}(\mathcal{Q}_F) > 0$. Explicitely, one may choose any $k > k_0 := \frac{-\mu_{\beta_R, \max}(\mathcal{F})}{\mu_{\alpha_R, \min}(\mathcal{F})}$.*

Proof. Indeed, by assumption, $\mu_{\alpha_R}(\mathcal{Q}_F) \geq \mu_{\alpha_R, \min}(\mathcal{F}) > 0$ for any such \mathcal{Q}_F . \square

The main observation is the following

Lemma 4.3. *Either $\det(\mathcal{Q}_G) \geq (\tau \circ r)^*(\det(\mathcal{Q}'_G))$ for some sheaf \mathcal{Q}'_G on W , or $\det(\mathcal{Q}_G)$ is an effective non-zero divisor when restricted to a generic fibre of $(\tau \circ r) : R \rightarrow W$. (The symbol $A \leq B$ between divisors means that $B - A$ is effective).*

Before proving this lemma, let us show that it implies Theorem 4.1: in the first case, we have: $\det(\mathcal{Q}_G) = (\tau \circ r)^*(\det(\mathcal{Q}'_G))$, and thus $\mu_{\alpha_R}(\mathcal{Q}_G) = 0$, since $(\tau \circ r)_*(\alpha_R) = 0$. Hence: $\mu_{\gamma_R}(\mathcal{Q}_G) = \mu_{\beta_R}(\mathcal{Q}_G) \geq \mu_{\beta_R, \min}(\mathcal{G}) > 0$. In the second case, we have, by the bigness of α_R on the fibres of $\tau \circ r$, and the fact that $(\tau \circ r)_*(\alpha_R) = 0$: $\mu_{\alpha_R}(\det(\mathcal{Q}_G)) > 0$. Thus $\mu_{k \cdot \alpha_R + \beta_R}(\mathcal{Q}_G) > 0$ if $k > 0$ is sufficiently large (once \mathcal{Q} is chosen to minimise μ_{β_R} among quotients of \mathcal{E}). \square

Proof. (of lemma 4.3) Let thus $\mathcal{M} \subset (\tau \circ r)^*(\mathcal{G})$ be a subsheaf of rank $t > 0$, and let $t' \leq t$ be the rank of the direct image sheaf $\mathcal{M}' := (\tau \circ r)_*(\mathcal{M})$. If $t = t'$, we are in the first case, and in the second case if $t' < t$. Indeed: in the first case, $((\tau \circ r)^*(\mathcal{M}')^{sat}/\mathcal{M})$ is torsion, and $((\tau \circ r)^*(\mathcal{M}'))^{sat} = (\tau \circ r)^*((\mathcal{M}')^{sat})$, the saturations being taken in $(\tau \circ r)^*(\mathcal{G})$ and in \mathcal{G} respectively. We thus have (after taking intersections with the relevant movable classes): $\det(\mathcal{M}) \leq (\tau \circ r)^*(\det(\mathcal{M}')^{sat})$ in this case.

In the second case, we consider the natural rational map $\varphi : R \rightarrow (\tau \circ r)^*(Grass(t, \mathcal{G}))$, which we may assume to be regular (by modifying suitably R). The property $t' < t$ means that the image of the generic fibre R_w of $\tau \circ r$ by φ is positive-dimensional. We have: $\mathcal{Q}_G = \varphi^*(Univ)$, and thus $\det(\mathcal{Q}_G) = \varphi^*(\det(Univ))$, if $Univ \rightarrow Grass(t, \mathcal{G})$ is the universal bundle of rank t on $Grass(t, \mathcal{G})$. The assertion in this second case thus immediately follows from the fact that $\det(Univ)$ is ample on the fibres of $Grass(t, \mathcal{G}) \rightarrow W$, and that the fibres of $\varphi(R)$ over W are positive-dimensional. \square

4.2. Case of a non-saturated quotient. We now consider the same situation $R, T, Y, W, W'', \alpha_R, \beta_R, \beta, \mathcal{F}, \mathcal{E}, \mathcal{G}$ as before, but assume instead that the exact sequence we have is:

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{H} \rightarrow 0,$$

where $\mathcal{H} \subset (\tau \circ r)^*(\mathcal{G})$ is a sheaf such that the quotient $(\tau \circ r)^*(\mathcal{G})/\mathcal{H}$ is of torsion, with support contained in a divisor $F \subset R$ such that $(\tau \circ r)(F) \subsetneq W$. In particular, we thus have: $\alpha_R \cdot \det(\mathcal{Q}_{\mathcal{H}}) = \alpha_R \cdot (\det(\mathcal{Q}_{\mathcal{G}}))$ if $\mathcal{Q}_{\mathcal{H}} \subset \mathcal{Q}_{\mathcal{G}}$ are quotients of $(\tau \circ r)^*(\mathcal{G})$ such that the cokernel has support in F . We get the following strengthening of Theorem 4.1, under an additional condition on β_R :

Corollary 4.4. *Assume that $\beta_R = \beta'_R + k' \alpha_R$, where $k' > 0$ is real, and $\beta'_R \cdot F_j = 0$, for any component F_j of F . The conclusion of Theorem 4.1 then still holds for \mathcal{E} . (ie: $\mu_{k \cdot \alpha_R + \beta_R, \min}(\mathcal{E}) > 0$ if $k > 0$ is sufficiently large).*

Proof. Any quotient of \mathcal{H} injects into a quotient of \mathcal{G} , with cokernel with support in F . The determinants of these quotients thus differ by a divisor supported on F , which has the same intersection with β_R and β'_R by assumption made that $(\tau \circ r)(F) \subsetneq W$. \square

4.3. Equivariant version. The diagram we consider is now:

$$\begin{array}{ccccccc} R & \xrightarrow{r} & T & \xrightarrow{\rho} & Y & \xrightarrow{\pi} & X \\ & & \downarrow \tau & \searrow \sigma & & & \downarrow f \\ W'' & \xrightarrow{s} & W' & \xrightarrow{p'} & W & \xrightarrow{p} & Z \end{array}$$

We assume here that both $\pi : Y \rightarrow X$, and $p : W \rightarrow Z$ are finite, Galois, of groups G and H respectively, and that T is a component of the normalisation of $Y \times_Z W$, with projections $\rho : T \rightarrow Y$ and $\sigma : T \rightarrow W$. Thus T is naturally L -Galois, equipped with an action of $L \subset G \times H$ such that ρ is finite, Galois, with group $H' \subset H$. Notice that L is surjective on both G and H , since T surjects finitely on both Y and $X_W := X \times_Z W$, which is irreducible because so are the generic fibres of f , and so also those of $f_W : X_W \rightarrow W$, so that every component of: $Y \times_Z W = Y \times_X (X \times_Z W) = Y \times_X X_W$ projects surjectively, not only on Y , but also on X_W .

The composition $\sigma = p' \circ \tau$ of the diagram is the Stein factorisation of σ , with τ connected and p' Galois, finite, of group G' , the kernel of the natural projection $L \rightarrow G'$, also the subgroup of G preserving the fibres of τ .

We assume that r is an L -equivariant resolution of T , and that s is a G' -equivariant resolution of W' .

We then also consider locally free sheaves \mathcal{E} on Y and \mathcal{G} on W , \mathcal{E} (resp. \mathcal{G}) being G -invariant (resp. H -invariant). Their liftings to R will be denoted by \mathcal{E}_R and \mathcal{G}_R , respectively.

We moreover assume that there exists an L -equivariant sheaf morphism $\Delta : \mathcal{E}_R \rightarrow \mathcal{G}_R$ with image $\mathcal{H} \subset \mathcal{G}_R$ such that the quotient $\mathcal{G}_R/\mathcal{H}$ is torsion on R , with (L -invariant) support contained in a divisor $F_R = (\pi \circ \rho \circ r)^{-1}(F)$, $F \subset X$ a divisor such that $f(F) \subsetneq Z$. Since Δ is L -equivariant, we see that $\mathcal{F}_R := \text{Ker}(\Delta) = (\pi \circ \rho \circ r)^*(\mathcal{F})$, for some G -invariant subsheaf $\mathcal{F} \subset \mathcal{E}$.

We assume moreover that we have movable classes α, β' on X , and β on Z such that:

1. $f_*(\alpha) = 0$, $f_*(\beta') = \beta$.
2. α is 'big' on the general fibre X_z of f .
3. There exists $k > 0$ such that $\beta' = \beta'' + k.\alpha$, where β'' is movable, and $\beta''.F_j = 0$, for every component F_j of F .
4. $\mu_{\alpha_Y, \min}(\mathcal{F}) > 0$.
5. $\mu_{\beta_W, \min}(\mathcal{G}) > 0$.

Here α_Y, β_W denote the liftings to Y and W of α and β respectively. (We adopt this notation for liftings, whenever defined, to any space in the diagram using the given maps. For example: $\beta'_Y := \pi^*(\beta')$, but β'_Y is not the lifting of β in any natural sense).

The equivariant version we shall need is the following:

Corollary 4.5. *In the above situation, and under the above hypothesis, we have, for any sufficiently large real number $k > 0$: $\mu_{\gamma, \min}(\mathcal{E}) > 0$, if $\gamma := k.\alpha_Y + \beta_Y$.*

Proof. It mainly consists in lifting up and down the classes α, β', β , and checking the above properties 1,2,3,4,5, together with the equivariance conditions at the levels of R, T, Y, W' , and in applying the arguments of Theorems 4.1 and corollary 4.4, in order to get the conclusion.

Lemma 4.6. *The properties 1,2,3 above imply:*

1. $(\tau \circ r)_*(\alpha_R) := \alpha_{W'} = 0$,
- 1'. $(\tau \circ r)_*(\beta'_R) := \beta'_{W'} = (p \circ p')^*(\beta)$;
2. α_R is 'big' on the general fibre $R_{w'}$ of $\tau \circ r : R \rightarrow W'$.
3. Let $\beta''_R := \beta'_R - k.\alpha_R$. Then: $(\beta''_R).F_j^R = 0$, for each component F_j^R of $F^R := (\pi \circ \rho \circ r)^{-1}(F)$.

Proof. All of these are easily obtained from the projection formula.

1. Let $E \subset W'$ be an irreducible divisor. Then (up to positive multiplicative integers, which are degrees of finite maps):

$$\begin{aligned} ((\tau \circ r)_*(\alpha_R)).E &= (\alpha_R).(\tau \circ r)^*(E) = ((\pi \circ \rho \circ r)^*(\alpha)).((\tau \circ r)^*(E)) \\ &= \alpha.[((\pi \circ \rho \circ r)_*((\tau \circ r)^*(E)))] = \alpha.[f^*((p \circ p')_*(E))] \\ &= (f_*(\alpha)).[(p \circ p')_*(E)] = 0, \forall E. \end{aligned}$$

1'. The sequence of equalities is exactly the same as before.

2. The map $(\pi \circ \rho \circ r) : R_{w'} \rightarrow X_z, z := (p \circ p')(w')$ is generically finite, and α_R on this fibre $R_{w'}$ is by definition, the lifting of α on X_z . This proves the assertion.

3. We have $(\tau \circ r)(F_j^R) \subset (p \circ p')^{-1}(f(F)) \subsetneq W'$, the assertion then follows from the same computation as for the assertion 1 above. \square

We denote by $\mathcal{F}_R, \mathcal{E}_R, \mathcal{H}_R, \mathcal{G}_{W''}$ the liftings of $\mathcal{F}, \mathcal{E}, \mathcal{G}, \mathcal{H}$ to R, R, R and W'' respectively.

Let us first observe that, by Lemma 5.1, we have the equalities:

$$\begin{aligned}\mu_{\alpha_R, \min}(\mathcal{E}_R) &= \mu_{\alpha_Y, \min}(\mathcal{E}) \\ \mu_{\alpha_R, \min}(\mathcal{F}_R) &= \mu_{\alpha_Y, \min}(\mathcal{F}) \\ \mu_{\beta_{W''}, \min}(\mathcal{G}_{W''}) &= \mu_{\beta_W, \min}(\mathcal{G})\end{aligned}$$

We can thus lift everything to R , and work there, using the same hypothesis for their liftings as for $\alpha, \beta, \beta', \beta''$, just taking into account the L -equivariance properties.

Let thus \mathcal{Q} be a $G \times H$ quotient of \mathcal{E}_R . It fits into an exact sequence

$$\mathcal{Q}_F \rightarrow \mathcal{Q} \rightarrow \mathcal{Q}_H \rightarrow 0,$$

for $G \times H$ -invariant sequence of sheaves induced by $(\rho \circ r)^*(\Delta)$, which is an equivariant map of sheaves on R (since Δ is G -equivariant).

The same argument as in Lemma 4.2 shows that for some explicit k_0 , $\mu_{k' \cdot \alpha_R + \beta'_R}(\mathcal{Q}_F) > 0$ if $k > k_0$. By the properties 1' and 3 of the preceding Lemma 4.6, we have, for every $k > 0$:

$$\mu_{k \cdot \alpha_R + \beta'_R}(\mathcal{Q}_H) = \mu_{(k+k') \cdot \alpha_R + \beta''_R}(\mathcal{Q}_H) \geq \mu_{k \cdot \alpha_R + \beta'_R, \min}(\mathcal{G}) > 0.$$

Now, there are, as in Lemma 4.3, two cases concerning \mathcal{Q}_H : by the property 1 of the preceding Lemma 4.6: either $\text{rank}((\tau \circ r)_*(\mathcal{Q}_H)) = \text{rank}(\mathcal{Q}_H)$, or $\text{rank}((\tau \circ r)_*(\mathcal{Q}_H)) < \text{rank}(\mathcal{Q}_H)$. Using Lemma 4.3, we conclude in the first case that $\mu_{k \cdot \alpha_R + \beta'_R}(\mathcal{Q}_H) \geq \mu_{\beta'_R, \min}(\mathcal{G}_R) > 0$. In the second case, we conclude from the same argument that $\mu_{k \cdot \alpha_R + \beta'_R}(\mathcal{Q}_H) > 0$ for $k > 0$ sufficiently large. \square

4.4. Case of an orbifold morphism. The diagram of the preceding section will be constructed from an orbifold morphism $f : (X, D) \rightarrow (Z, D_Z)$, in which (Z, D_Z) is the orbifold base of f , together with movable classes α, β' on X , and β on Z such that: $f_*(\alpha) = 0$, and $f_*(\beta') = \beta$.

Let $\pi : Y \rightarrow X$ and $p : W \rightarrow Z$ be Kawamata-covers adapted to (X, D) and (Z, D_Z) respectively, of Galois groups G, H .

Let $\mathcal{F} \subset \mathcal{E} := \pi^*(T(X, D))$ be the maximal destabilising subsheaf of $\mathcal{E} = \pi^*(T(X, D))$ relative to $\pi^*(\alpha)$. We assume that the associated fibration has f as 'neat' birational model.

Let T be a component of the normalisation of $Y \times_Z W$, together with the projections $\rho : T \rightarrow Y$ and $\sigma : T \rightarrow W$, so that $L \subset G \times H$ naturally acts on T . We take further an L -equivariant resolution $r : R \rightarrow T$ of T . In this way, the maps $\rho \circ r : R \rightarrow Y$ and $\tau \circ r : R \rightarrow W$ are Galois, of groups $H' \subset H$ and $G' \subset G$ respectively. We consider next $\sigma = \tau \circ p' : T \rightarrow W$ the Stein factorisation of σ as $\tau : T \rightarrow W'$ and

$p' : W' \rightarrow W$. It enjoys the equivariance properties of the diagram of the preceding section.

Let $\mathcal{G} := (\sigma \circ r)^*(p^*(T(Z, D_Z)))$, and $\Delta := (\rho \circ r)^*(\pi^*(df)) : \mathcal{E} \rightarrow \mathcal{G}$: this is an L -equivariant map by construction, its existence is guaranteed by Proposition 2.11. Let $\mathcal{H} := [\Delta(\mathcal{E})]$, we thus have an inclusion $\mathcal{H} \subset \mathcal{G}$, and an exact sequence:

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{H} \rightarrow 0.$$

Moreover, by Proposition 2.13, the cokernel sheaf \mathcal{G}/\mathcal{H} is torsion, with support in a divisor $F \subset X$ ‘partially supported on fibres of f ’ (See Definition 2.12). This property permits to show, by Proposition 3.6.(6), the existence of a movable class $\beta'' := \beta_X$ on X such that $\beta'' \cdot F_j = 0$ for all components F_j of F , and to obtain from Corollary 4.5:

Theorem 4.7. *In the preceding situation, assume also that $\mu_{\alpha_Y}^G(\mathcal{F}) > 0$, that $\mu_{p^*(\beta), \min}(p^*(T(Z, D_Z))) > 0$, and that α is ‘big’ on the ‘general’ fibre of f . Then $k \cdot \alpha + \beta_X := \gamma$ satisfies: $\mu_{\pi^*(\gamma), \min}(\pi^*(T(X, D))) > 0$, for any sufficiently large $k > 0$.*

5. HARDER-NARASIMHAN FILTRATIONS FOR ORBIFOLD BUNDLES

In this section, we establish the fact that the fibration on X constructed in [15] from an orbifold foliation of positive minimal slope relatively to a movable class α on X depends only on the orbifold pair (X, D) and α , and not on the Kawamata cover used to define it, and that moreover it is preserved under orbifold birational equivalence. These results are natural complements of those in [15]. The arguments lead to more general statements however.

We refer to [13] and [15] for the definitions and basic properties of slope, (semi-)stability, and Harder Narasimhan filtration of a (reflexive) sheaf \mathcal{E} , relative to a movable class α on a projective manifold. We used in [15] the following notion:

Conventions: Let $\pi : Y \rightarrow X$ be a generically finite morphism between two complex connected projective manifolds, which is Galois for a certain finite group G (which acts effectively and transitively on the generic fibres of g). Let α be a movable class on X . Then $\pi^*(\alpha)$ is still a movable class on Y .

Lemma 5.1. *Let $g : X' \rightarrow X$ be a surjective morphism between normal irreducible complex projective spaces. Let \mathcal{E} be a reflexive coherent sheaf on X , and $\mathcal{F}' \subset g^*(\mathcal{E})$ be a G -invariant saturated subsheaf. Denoting with $(\cdot)^{\text{sat}}$ the saturation inside $g^*(\mathcal{E})$, we have: :*

1. $\mathcal{F}' = g^*(\mathcal{F})^{\text{sat}}$ with $\mathcal{F} := g_*(\mathcal{F}')$ if g is birational.
2. $\mathcal{F}' = g^*(\mathcal{F})^{\text{sat}}$ if $\mathcal{F} = (g_*(\mathcal{F}'))^G$ if g is finite, Galois of group G , and if \mathcal{F}' is preserved by the action of G on $g^*(\mathcal{E})$. Here $(g_*(\mathcal{F}'))^G$ is the subsheaf of $g_*(g^*(\mathcal{E})) = \mathcal{E}$ generated by the presheaf of G -invariant

sections of \mathcal{F}' defined on open sets of X' of the form $g^{-1}(U)$, $U \subset X$ open. Moreover, the support of $(g^*(\mathcal{F})^{sat}/g^*(\mathcal{F}))$ is contained in the codimension at least 2 subset $S := g^{-1}(\text{Sing}(X) \cup g(\text{Sing}(X')))$ of X' .

Proof. Assume first that g is birational.

The evaluation map $v : g^*(g_*(\mathcal{F}')) \rightarrow \mathcal{F}$ is then isomorphic over the generic point of X , and $\mathcal{F} := g_*(\mathcal{F}') \subset \mathcal{E} := g_*(g^*(\mathcal{E}))^G$. Since \mathcal{F}' is saturated in $g^*(\mathcal{E})$, we obtain the given isomorphism.

Assume now that we are in the second situation. By its definition, the sheaf $\mathcal{F} := (g_*(\mathcal{F}'))^G \subset \mathcal{E}$ is generated locally by sections of the form $s := \frac{1}{N} \cdot (\sum_{h \in G} h^*(s'))$, for s' a local section of \mathcal{F}' . Thus it is locally of finite type, and hence coherent as an \mathcal{O}_X -module. From the flatness of g outside of S , we deduce that $\mathcal{F}' = g^*(\mathcal{F}) = g^*(\mathcal{F})^{sat}$ there. We conclude using the evaluation map $v : g^*(\mathcal{F}) \rightarrow \mathcal{F}'$. \square

Corollary 5.2. *Assume that $g : X' \rightarrow X$ is as in 5 above, with X, X' smooth. Assume that $g = h \circ k$, with $h : Y \rightarrow X$ finite, Galois, of group G , and $k : X' \rightarrow Y$ birational and G -equivariant. Let α be a movable class on X , and $g^*(\alpha) := \alpha'$ be its inverse image on X' . Let \mathcal{E} be a reflexive sheaf on X . Then $\mu_{\alpha, \min}(\mathcal{E}) = \mu_{\alpha', \min}(g^*(\mathcal{E}))$, and the $g^*(\alpha)$ -maximal destabilising sheaf \mathcal{F}'_{max} of $g^*(\mathcal{E})$ is equal to $g^*(\mathcal{F})^{sat}$, where \mathcal{F} is the α -maximal destabilising sheaf of \mathcal{E} for α . More generally, the Harder-Narasimhan filtration $HN_{g^*(\alpha)}(g^*(\mathcal{E}))$ is equal to the saturation of $g^*(HN_{\alpha}(\mathcal{E}))$ in $g^*(\mathcal{E})$, with equality of corresponding slopes.*

Moreover, if $\mathcal{E}' \subset g^(\mathcal{E})$ is a G -invariant subsheaf such that $\mathcal{E}'^{sat} = g^*(\mathcal{E})$, then: $HN_{g^*(\alpha)}(\mathcal{E}')^{sat} = g^*(W^\bullet(\mathcal{E}', \alpha))$, for some suitable, uniquely defined, filtration $W^\bullet(\mathcal{E}', \alpha)$ on \mathcal{E} .*

Conversely, $HN_{g^(\alpha)}(\mathcal{E}') = g^*(W^\bullet(\mathcal{E}', \alpha)) \cap \mathcal{E}'$.*

Proof. Let $\mathcal{F}' \subset g^*(\mathcal{E})$ be any G -invariant saturated subsheaf. Define $\mathcal{F}_Y := k_*(\mathcal{F}')^{sat}$ (saturation in $h^*(\mathcal{E})$ here), and $\mathcal{F} := h_*(\mathcal{F}_Y) = h_*(\mathcal{F}_Y)^{sat}$. All of these sheaves are G -invariant. From the preceding lemma, we obtain that $\mathcal{F}' = g^*(\mathcal{F})^{sat}$. Hence, using the last assertion of the preceding lemma, we obtain: $\det(\mathcal{F}') = g^*(\det(\mathcal{F})) + E$, where E is a g -exceptional divisor on X' (i.e: all of its components are mapped in codimension at least 2 in X by g). Thus: $g^*(\alpha) \cdot \det(\mathcal{F}') = \alpha \cdot \det(\mathcal{F})$, since $g^*(\alpha) \cdot E = 0$.

If we apply this to the $g^*(\alpha)$ -maximal destabilising subsheaf \mathcal{F}'_{max} of $g^*(\mathcal{E})$, we obtain that $\mathcal{F}' = g^*(\mathcal{F})^{sat}$, if \mathcal{F} is the α -maximal destabilising subsheaf of \mathcal{E} . The statements concerning the Harder-Narasimhan filtrations, the slopes, and $\mu_{g^*(\alpha), \min}(g^*(\mathcal{E}))$ follow immediately. The last statement is proved in the same way, since the Harder-Narasimhan filtration of \mathcal{E}' relatively to $g^*(\alpha)$ is G -invariant. \square

We shall need a slight generalisation, too:

Corollary 5.3. *Let $g : X' \rightarrow X$, $G, Y, \mathcal{E}, \mathcal{E}' = g^*(\mathcal{E}), \alpha$ be as in 5 above.*

Let moreover $h' : X'' \rightarrow X'$ be birational, G -equivariant, with X'' smooth. Let $\mathcal{E}'' \subset (g \circ h')^*(\mathcal{E}) := (h')^*(\mathcal{E}')$ be a reflexive G -invariant subsheaf such that the support of $((h')^*(\mathcal{E}')/\mathcal{E}'')$ is contained in the exceptional divisor of h' . The conclusions of the preceding corollary then hold for \mathcal{E}'' and $\alpha'' := (g \circ h')^*(\alpha)$ in place of $g^*(\mathcal{E})$ and $g^*(\alpha)$. More precisely, we have: $HN_{(g \circ h')^*(\alpha)}^G(\mathcal{E}'') = (g \circ h')^*((HN_\alpha(\mathcal{E}))^{sat}) \cap \mathcal{E}''$, with equality of corresponding slopes.

Proof. The conclusions of the preceding lemma apply to $(g \circ h')^*(\mathcal{E})$. The $(g \circ h')^*(\alpha)$ -maximal destabilising G -subsheaf \mathcal{F}'' of \mathcal{E}'' is the intersection with \mathcal{E}'' of the $(g \circ h')^*(\alpha)$ -maximal destabilising G -subsheaf \mathcal{F}^+ of $(g \circ h')^*(\mathcal{E})$, and so its $(g \circ h')^*(\alpha)$ -slope is the same as the one of \mathcal{F}^+ , by the assumption on the support of $((h')^*(\mathcal{E}')/\mathcal{E}'')$. The other assertions follow immediately. \square

5.1. Independence of the adapted cover. Let (X, D) be, as always here, a smooth projective orbifold pair, and let $\pi : Y \rightarrow X$ a Kawamata cover (see [15], §5) adapted to (X, D) . Recall that Y is smooth, π G -Galois, Kummer and finite, for some Abelian group G . Moreover, the locally free sheaves $\pi^*(T(X, D))$ and its dual $\pi^*(\Omega^1(X, D))$ are defined on Y . More precisely, if $D = \sum_j c_j D_j$, with $c_j = \frac{a_j}{b_j}$ as above, then π ramifies to a certain order $m > 0$, divisible by each of the b_j 's over $\text{Supp}(D) = \cup_j D_j$ (and also over some additional components H needed to construct π globally).

Let α be a movable class on X , we thus get $HN_{\pi^*(\alpha)}^G(\pi^*(T(X, D)))$, and it is proved in [15], that the saturation $HN_{\pi^*(\alpha)}^G(\pi^*(T(X, D)))^{sat}$ inside $\pi^*(TX)$ is of the form $\pi^*(W^\bullet(D, \alpha, \pi))$ for a certain filtration W^\bullet on TX (which is, of course, not $HN_\alpha(TX)$ in general, if $D \neq 0$).

Our aim here is to prove that the filtration $W^\bullet(D, \alpha, \pi)$ is in fact¹¹ independent of the adapted cover π and that it only depends on D and α . It will thus simply be written as $W^\bullet(D, \alpha)$ in the sequel. This permits, once π is given, to reconstruct $HN_{\pi^*(\alpha)}(\pi^*(T(X, D)))$ by simply intersecting $\pi^*(W^\bullet(D, \alpha))$ with $\pi^*(T(X, D))$.

So we consider a second Kawamata cover $\pi' : Y' \rightarrow X$ adapted to (X, D) , of group $G' = \text{Gal}(Y'/X)$. Let

$$(13) \quad Z \subset (Y' \times_X Y)^n$$

be any component of the normalisation of fibered product, together with the natural projections $q : Z \rightarrow X$, $p : Z \rightarrow Y$ and $p' : Z \rightarrow Y'$. It is acted on by its normal stabilizer $L \subset G' \times G$, which is onto on both G and G' . Indeed: Z is surjective and finite on both Y' and Y , which are irreducible. Moreover, q is L Galois. The projections p and p' are respectively H' -Galois (resp. H -Galois) for some subgroups H' (resp.

¹¹In accordance with the ‘orbifold’ nature of these notions.

$H)$ of G' (resp. G). The components of $(Y \times_Z W)^n$ are exchanged under the action of $(G \times G')/L$.

Let finally $r : Z' \rightarrow Z$ be an L -equivariant resolution of Z . Let $q := \pi \circ p \circ r = \pi' \circ p' \circ r$.
of $(G \times H)/L$.

Theorem 5.4. *We have:*

1. $p'^*(\pi'^*(T(X, D))) = p^*(\pi^*(T(X, D))) := T$.
2. $HN_{q^*(\alpha)}(T) = (p \circ r)^*(HN_{\pi^*(\alpha)}(\pi^*(T(X, D))))^{sat} = (p' \circ r)^*(HN_{\pi'^*(\alpha)}(\pi'^*(T(X, D))))^{sat}$.
3. *We have equality of the corresponding slopes in the above filtrations.*

Proof. The first equality is easily computed outside of $S \subset X$ consisting of the union of $Sing(\text{Supp}(D))$ and of the intersection of $\text{Supp}(D)$ with the union of all the additional components H of the ramification loci of π and π' , since it just consist in liftinging ‘fractional’ vector fields of the form $x^c \cdot \partial x$, $c \in \mathbb{Q} \cap]0, 1]$, under maps $x = z^{m \cdot m'} = z^{m'/m}$ for integers m, m' such that mc and $m'c$ are integers. Since both $\pi^*(T(X, D))$ and $\pi'^*(T(X, D))$ are locally free, this equality extends to Z , and then to Z' .

The last two statements are then immediate applications of Corollary 5.2. \square

Corollary 5.5. *For given (X, D) , π, G and α as above, the filtration $W^\bullet(D, \alpha)$ on TX such that $\pi^*(W^\bullet(D, \alpha)) = HN_{\pi^*(\alpha)}(\pi^*(T(X, D)))^{sat}$, the saturation being in $\pi^*(TX)$, is independent on π .*

Proof. Indeed, for r, π', p', p as above, the saturations in $(\pi \circ p \circ r)^*(TX)$ of $(p \circ r)^*(HN_{\pi^*(\alpha)}(\pi^*(T(X, D))))$ and $(p' \circ r)^*(HN_{\pi'^*(\alpha)}(\pi'^*(T(X, D))))$ coincide with that of $HN_{q^*(\alpha)}(T)$. \square

Assume now that $\alpha \cdot (K_X + D) < 0$. Let $\mathcal{F} \subset \pi^*(T(X, D))$ be the maximal destabilising subsheaf: it has a positive α -slope. It is proved in [15] that its saturation in $\pi^*(TX)$ is of the form $\pi^*(\mathcal{F}_X)$ for some algebraic foliation \mathcal{F}_X on X , with $\mathcal{F}_X = \text{Ker}(df)$ for some rational fibration $f : X \dashrightarrow Z$. From Corollary 5.5, we obtain:

Corollary 5.6. *The foliation \mathcal{F}_X and the fibration f (up to birational equivalence) depend on D and α , but not on π .*

5.2. Birational orbifold invariance. We shall apply the preceding Proposition 2.11 together with the Corollary 5.3 in order to show the birational orbifold invariance of the equivariant Harder-Narasimhan filtration of $\pi^*(T(X, D))$ relative to a movable class α .

The situation is the following: Let $f(X', D') \rightarrow (X, D)$ be a birational orbifold morphism between smooth projective complex orbifold pairs. We thus have, in particular: $f_*(D') = D$.

Let $\pi : Y \rightarrow (X, D)$ and $\pi' : Y' \rightarrow (X', D')$ be adapted Kawamata-covers of groups G, G' . We consider, as before, an irreducible component $T \subset (Y' \times_X Y)^n$ of its normalised fibre-product $(Y' \times_X Y)^n$ over X , together with its projections $\rho : T \rightarrow Y$ and $\rho' : T \rightarrow Y'$. We have a natural action of $L \subset G' \times G$ on T . We also choose an L -equivariant resolution of singularities $r : R \rightarrow T$ for this action.

Define two locally-free sheaves $\mathcal{E}' := (\rho' \circ r)^*((\pi')^*(T(X', D')))$ and $\mathcal{E} := (\rho \circ r)^*((\pi)^*(T(X, D)))$ on R . By the previous Proposition 2.11, we get a sheaf homomorphism $(\pi' \circ \rho' \circ r)^*(df) : \mathcal{E}' \rightarrow \mathcal{E}$, since f is an orbifold morphism.

Lemma 5.7. *The cokernel of $(\pi' \circ \rho' \circ r)^*(df) : \mathcal{E}' \rightarrow \mathcal{E}$ is contained in the exceptional divisor of $(\rho \circ r) : R \rightarrow Y$.*

Proof. This follows from the fact that $f_*(D') = D$ by the same argument used to prove Theorem 5.4, which shows that the sheaves \mathcal{E} and \mathcal{E}' coincide in codimension one outside of the inverse image in R of the exceptional divisor of f in X' . \square

Applying now Corollary 5.3 to $\mathcal{E}', \mathcal{E}$ and to $\rho \circ r : T \rightarrow Y$, writing $h = (f \circ \pi' \circ \rho' \circ r) = (\pi \circ \rho \circ r) : R \rightarrow X$, we get:

Proposition 5.8. *Let $f : (X', D') \rightarrow (X, D)$ be an orbifold birational morphism as above. Then, taking saturation inside \mathcal{E} , we have:*

$$[(\rho' \circ r)^*(HN_{(f \circ \pi')^*(\alpha)}^{G'}(\pi'^*(T(X', D'))))]^{sat} = [(\rho \circ r)^*(HN_{\pi^*(\alpha)}^G(\pi^*(T(X, D))))]^{sat}$$

We now have two filtrations $W^\bullet(D, \alpha)$ on TX and $W^\bullet(D', \alpha')$ on TX' , with $\alpha' := f^*(\alpha)$, obtained by descending the Galois-invariant Harder-Narasimhan filtrations of $\pi'^*(T(X', D'))$ relative to $(f \circ \pi')^*(\alpha)$, and similarly for (X, D) and $\pi^*(\alpha)$. See Theorem 5.4 and Corollary 5.5

Corollary 5.9. *The situation being the same as in 5.8, we have: $W^\bullet(D', \alpha') = [f^*(W^\bullet(D, \alpha))] \cap TX'$.*

In other words: these filtrations are birationally invariant and independent on the Kawamata-covers, with respect to liftings of the given movable class.

Remark 5.10. *When a fibration $f : (X, D) \rightarrow (Z, D_Z)$ (possibly birational) is given, together with a Kawamata cover $p : W \rightarrow Z$ adapted to (Z, D_Z) , it is always possible to construct, after suitable blow-ups of (X, D) , some (X', D') , and a Kawamata cover $\pi' : Y' \rightarrow (X', D')$ adapted to D' , such that one has a regular map $f' : Y' \rightarrow W$ with: $f \circ \pi' = p' \circ f' : X' \rightarrow Z$. This permits to simplify the diagrams to come, by choosing $R = T = Y'$, but at the (small) expense of not taking an arbitrary $\pi : Y \rightarrow (X, D)$.*

6. PROOF OF THEOREM 1.1

Before starting the proof, let us observe that the 3 properties of the statement are ‘up’-birationally invariant¹² in the sense that if they hold on (X, D) , they also hold on (X', D') for any birational $f : X' \rightarrow X$ which induces an orbifold morphism $f : (X', D') \rightarrow (X, D)$ (see Definition 2.1), the movable classes involved being always the inverse images of the initial ones.

Proof. We shall show the implications $3 \implies 2 \implies 1 \implies 3$.

• $3 \implies 2$. This is just a consequence of the fact that $h^0(X, \mathcal{F} \otimes A) = 0$ if $\mu_{\alpha, \max}(\mathcal{F}) < \alpha.A$ for some movable class α . Now by our hypothesis:

$$\mu_{\alpha, \max}(\otimes^m(\Omega^1(X, D))) = -m.\mu_{\alpha, \min}(T(X, D)) < \alpha.A,$$

if $m > m(A) := \frac{\alpha.A}{\mu_{\alpha, \min}(T(X, D))}$.

• $2 \implies 2'$. Obvious.

• $2' \implies 1$. Let any fibration which is a ‘neat’ orbifold morphism to its orbifold base $f : (X, D) \rightarrow (Z, D_Z)$, with $\dim(Z) > 0$, be given.

Let us consider again the diagram considered in §4.3, with the same meaning as in §4.4:

$$\begin{array}{ccccccc} R & \xrightarrow{r} & T & \xrightarrow{\rho} & Y & \xrightarrow{\pi} & X \\ & & \downarrow \tau & \searrow \sigma & & & \downarrow f \\ W'' & \xrightarrow{s} & W' & \xrightarrow{p'} & W & \xrightarrow{p} & Z \end{array}$$

The elements of $H^0(Z, (K_Z + D_Z)^{\otimes m} \otimes B)$ lift injectively to R as inverse images of G -invariant elements of $H^0(Y, \text{Sym}^m(\pi^*(\Omega^1(X, D))) \otimes \pi^*(f^*(B)))$. As all of these last ones vanish by assumption, so do the first ones. But this means precisely that $(K_Z + D_Z)$ is not pseudo-effective.

• $1 \implies 3$. This is the actual content of Theorem 1.1, and its most involved part, requiring the preliminaries above. The proof will work by induction on $n := \dim(X)$.

We start with $n = 1$, so that X is a curve, $K_X + D$ is not pseudo effective, and there is only one non-zero movable class α up to non-zero homothety. Then $\pi^*(\Omega^1(X, D)) = \pi^*(K_X + D)$, and clearly: $\mu_{\alpha, \min}(\pi^*(X, D)) = -d.\alpha.(K_X + D) > 0$, if $d > 0$ is the geometric degree of π .

We thus assume that $n \geq 2$, and that the implication $1 \implies 3$ holds whenever $\dim(X) < n$.

We shall produce a fibration $f : (X, D) \rightarrow (Z, D_Z)$ with $\dim(Z) > 0$, together with suitable classes $\alpha \in \text{Mov}^0(X/Z)$ and $\beta \in \text{Mov}^0(Z)$ and $\beta_X \in \text{Mov}(X)$ such that $f_*(\beta_X) = \beta$, so as to be in position to apply

¹² ‘Up’ because this works under blow-ups, not under blow-downs in general.

the results in §4.4 to conclude that $\gamma := k.\alpha + \beta_X$ satisfies the property $\mu_{\pi^*(\gamma), \min}(T(X, D)) > 0$ if $k > 0$ is sufficiently large.

We assume that $\pi : Y \rightarrow X$ is a Kawamata cover adapted to (X, D) . Let $\alpha \in \text{Mov}(X)$ be any class such that $\alpha.(K_X + D) < 0$. Let $\mathcal{F} \subset \pi^*(T(X, D))$ be the (G -invariant) α -maximal destabilising subsheaf. It defines (by [15]) a rational fibration $f_0 : X_0 \dashrightarrow Z_0$ with $\mathcal{F}_{X_0} = \text{Ker}(df_0)$, where $\pi^*(\mathcal{F}_{X_0}) = \mathcal{F}^{\text{sat}}$, the saturation inside $\pi^*(TX)$. We write here $X_0 = X, Z_0 = Z$, because we shall now consider (new) suitable birational models X, Z of X_0, Z_0 . Moreover, this fibration is non-trivial: $\dim(Z) < n$. We put $d := n - \dim(Z) > 0$.

We now chose a (new) ‘neat’ birational model $f : (X, D) \rightarrow (Z, D_Z)$ of our original $f_0 : X_0 \dashrightarrow Z_0$, and a corresponding Kawamata cover of (X, D) . By Proposition 5.8, the G -invariant Harder-Narasimhan fibration of the new orbifold cotangent sheaf is the inverse image of the initial one. We can thus assume from the very beginning that f was a ‘strictly’ neat orbifold morphism, in the sense that it is neat, and that, moreover, any f -exceptional divisor $F \subset X$ is u -exceptional for the birational map $u : X \rightarrow X_0$, and $\alpha = u^*(\alpha_0)$. This ‘strictness’ property is obtained by first flattening the map f_0 .

Moreover, we have (using the notations of [15]): $0 > -\alpha.(\det(\mathcal{F})) = \alpha.(K_{X/Z} + D^{\text{hor}} - D(f))$, so that $K_{X_z} + D_z := (K_{X/Z} + D)|_{X_z}$ is not pseudo-effective for $z \in Z$ generic. There thus exists $\alpha' \in \text{Mov}^0(X/Z)$ such that $\alpha'.(K_{X/Z} + D) < 0$. Let us choose any of these. Let $\mathcal{F}' \subset \pi^*(T(X, D))$ be the (G -invariant) α' -maximal destabilising subsheaf.

Lemma 6.1. *In this situation, we have:*

1. $\mathcal{F}' \subset \mathcal{F}$.
2. \mathcal{F}' is an orbifold foliation on (X, D) . It is thus algebraic, too.

Proof. 1. Since $\mathcal{F} = \text{Ker}(\pi^*(df) : \pi^*(T(X, D)) \rightarrow (f \circ \pi)^*(TZ))$, it is sufficient to show that $\text{Hom}(\mathcal{F}', (f \circ \pi)^*(TZ)) = 0$, and thus, that $\mu_{\alpha', \min}(\mathcal{F}') = \mu_{\alpha'}(\mathcal{F}') > \mu_{\alpha', \max}((f \circ \pi)^*(TZ))$. By assumption, $\mu_{\alpha'}(\mathcal{F}') > 0$, since $\alpha'.(K_{X/Z} + D) < 0$, and \mathcal{F}' is the maximal α' -destabilizing subsheaf. On the other hand, , since $f_*(\alpha') = 0$, we have: $\mu_{\alpha', \max}(f \circ \pi)^*(TZ) \leq 0$, by Lemma 4.3, since the image of \mathcal{F}' , if nonzero, were a subsheaf of $(f \circ \pi)^*(TZ)$, and should be, either generically coming from from TZ , or having an anti-effective determinant along the fibres of τ .

2. The fact that \mathcal{F}' is an (X, D) -foliation in the sense of [15] is the usual slope-argument (going back to Y. Miyaoka), since \mathcal{F}' is maximal-destabilizing with positive slope. The algebraicity statement is one of the main results of [15]. \square

The next statements are immediate consequences of Lemma 6.1:

Corollary 6.2. *Assume $d_\alpha := d$ is minimal among all these d_α , $\alpha \in \text{Mov}(X)$ such that: $\alpha.(K_X + D) < 0$. Then:*

1. $d > 0$.
2. $\mathcal{F}' = \mathcal{F}$.
3. $\mu_{\alpha', \min}(\mathcal{F}) > 0$.

Remark 6.3. *This corollary applies in a relative setting $f : (X, D) \rightarrow Y$ as well, when $K_{\mathcal{F}}$ is not pseudo-effective, by just considering classes α' in $Mov(X/Y)$ such that $K_{\mathcal{F}_D} \cdot \alpha' < 0$, where $\mathcal{F}_D := \pi^*(Ker(df)) \cap \pi^*(T(X, D))$.*

We shall now fix such a class $\alpha' \in Mov^0(X/Z)$, and denote it α .

By assumption, for all fibrations $g : Z \dashrightarrow Y$ with $\dim(Y) > 0$, we have: $K_Y + D_Y$ is not pseudo-effective (replacing the initial g by a ‘neat’ birational model which induces an orbifold morphism $g : (Z, D_Z) \rightarrow (Y, D_Y)$ to its orbifold base (Y, D_Y)). Indeed, by the property 2.5 recalled from [9], the orbifold base of $g \circ f : (X, D) \rightarrow Y$ is also the orbifold base of $g : (Z, D_Z) \rightarrow Y$, because $f : (X, D) \rightarrow (Z, D_Z)$ is an orbifold morphism.

We have two cases: either $\dim(Z) = 0$, and the property 3 of Theorem 1.1 is established, or $n > \dim(Z) > 0$, and we can apply induction: the property 3 is then satisfied by (Z, D_Z) . There thus exists $\beta \in Mov^0(Z)$ such that $\mu_{\beta, \min}(p^*(T(Z, D_Z))) > 0$, if $p : W \rightarrow Z$ is any Kawamata cover adapted to (Z, D_Z) .

We shall now show that we are in position to apply the results of §4.4 to the classes α and β thus constructed. We thus consider again the diagram introduced in §4.4 and §4.3, with the same meaning:

$$\begin{array}{ccccccc}
 R & \xrightarrow{r} & T & \xrightarrow{\rho} & Y & \xrightarrow{\pi} & X \\
 & & \downarrow \tau & \searrow \sigma & & & \downarrow f \\
 W'' & \xrightarrow{s} & W' & \xrightarrow{p'} & W & \xrightarrow{p} & Z
 \end{array}$$

In order to apply Theorem 4.7, we thus choose the movable class $\beta_X = \beta''$ on X constructed in Proposition 3.6.(6). We thus have: $f_*(\beta'') = \beta$ and $\beta'' \cdot F_j = 0$ for all components F_j of the divisor F ‘partially supported on the fibres of f ’ which contains the support of the Cokernel $(\mathcal{G}/\mathcal{H})$ of the map Δ considered in the statement of Theorem 4.7, which thus applies and concludes the proof of Theorem 1.1.

We have thus proved the claimed conclusion 3 of Theorem 1.1 on some orbifold birational model $g : (X', D') \rightarrow (X, D)$ of (X, D) . The following lemma 6.4 shows that the conclusion still holds for (X, D) itself, when we apply this lemma to the natural inclusion $dg : T(X', D') \rightarrow T(X, D)$ lifted to a fibre-product of Kawamata covers adapted to them, as in Proposition 5.8.

Lemma 6.4. *Let $g : X' \rightarrow X$ be a birational morphism between two projective connected complex manifolds, let $\mathcal{E}' \rightarrow g^*(\mathcal{E})$ be an injection*

of torsionfree coherent sheaves on X' , which is generically isomorphic. Let $\alpha' \in \text{Mov}(X')$, and $\alpha := g_*(\alpha') \in \text{Mov}(X)$.

Then: $\mu_{\alpha', \min}(\mathcal{E}') \leq \mu_{\alpha, \min}(\mathcal{E})$.

In particular: $\mu_{\alpha, \min}(\mathcal{E}) > 0$ if $\mu_{\alpha', \min}(\mathcal{E}') > 0$.

Proof. Let \mathcal{Q} be any non-zero quotient of \mathcal{E} : it induces (by considering the intersection of the kernel corresponding to $g^*(\mathcal{Q})$ with \mathcal{E}' , and the related quotient) a quotient \mathcal{Q}' of \mathcal{E}' together with an injection $\mathcal{Q}' \rightarrow g^*(\mathcal{Q})$ which is generically isomorphic. We thus have: $\det(g^*(\mathcal{Q})) = \det(\mathcal{Q}') + E'$, where E' is an effective divisor supported on the exceptional divisor of g . We have: $\mu_{g^*(\alpha)}(g^*(\mathcal{Q})) = \mu_{\alpha}(\mathcal{Q}) \geq \mu_{\alpha, \min}(\mathcal{E})$. Moreover: $0 < \mu_{\alpha', \min}(\mathcal{E}') \leq \alpha'.\det(\mathcal{Q}') \leq \alpha'.\det(g^*(\mathcal{Q})) = \alpha.\det(\mathcal{Q})$, by the projection formula, which implies the claim. \square

The fact that α can be choosen to be ‘movable-big’ is a consequence of the following general:

Lemma 6.5. *Let \mathcal{E} be a coherent torsionfree sheaf on the connected complex projective manifold X . Let $\alpha, \beta \in \text{Mov}(X)$, with β big, be such that $\mu_{\alpha, \min}(\mathcal{E}) > 0$. Let $\alpha_t := \alpha + t\beta$. Then $\mu_{\alpha_t, \min}(\mathcal{E}) > 0$ for some $\varepsilon > 0$ and any $0 \leq t \leq \varepsilon$.*

Proof. The property is established in [13], Lemma 5.6 for α rational, and in [22], Theorem 3.4 in general when \mathcal{E} is α -stable. One deduces the general case by replacing \mathcal{E} by the successive α -semi-stable quotients of its α -Harder-Narasimhan filtration, and using Jordan-Hölder filtrations by stable sheaves of the same α -slope of these quotients. We thus get a filtration of \mathcal{E} by α -stable sheaves of positive slope. This property is then preserved for α_t , if $0 \leq t$ is sufficiently small. \square

\square

6.1. Relative version. The preceding result holds in a relative version as well, the proof being essentially similar.

Theorem 6.6. *Let $f : (X, D) \rightarrow Z$ be a surjective morphism of complex projective manifolds¹³ with connected fibres, (X, D) being a smooth orbifold pair. Assume that the general orbifold fibre (X_z, D_z) of f is sRC. Let $\pi : Y \rightarrow X$ be any Kawamata cover adapted to D .*

Then: $\mu_{\pi^(\alpha), \min}(\pi^*(T(X_z, D_z))) > 0$, for some $\alpha \in \text{Mov}^0(X/Z)$ ¹⁴.*

Proof. It is essentially a relative version of the proof of $1 \implies 3$ in the preceding section. We may, due to the lemma 6.5, replace (X, D) by an arbitrary orbifold modification. Because $K_{X_z} + D_z$ is not pseudo-effective, by assumption, there exists a class $\alpha \in \text{Mov}^0(X/Z)$ such that $(K_X + D).\alpha < 0$, and thus, taking the associated maximal destabilizing subsheaf G , and applying [15] to it, we get an algebraic foliation $g :$

¹³ Z being normal would actually suffice, here.

¹⁴See definition in §3.

$(X, D) \rightarrow Y$ over Z , with $d_\alpha := \dim(X_z) > 0$, and $\mu_{\pi^*(\alpha)}(G) > 0$. Choosing α as above such that d_α is minimal, we obtain also that $\mu_{\pi^*(\alpha), \min}(G) > 0$. If $d_\alpha := \dim(X_y) = \dim(X_z)$, we are finished. Notice that the claim thus holds true if $d := \dim(X_z) = 1$.

Otherwise, we argue by induction on $d := \dim(X_z) \geq 2$, assuming the claim to hold in strictly smaller relative dimensions. Let (Y, D_Y) be the orbifold base of $f : (X, D) \rightarrow Y$, and $h : Y \rightarrow Z$ be the factorisation morphism. By assumption $\dim(X) > \dim(Y) > \dim(Z)$, and the claim holds for both g and h , by the induction hypothesis. From the construction of g above, we have $\alpha \in \text{Mov}^0(X/Y)$ such that $\mu_{\pi^*(\alpha), \min}(F) > 0$, and from the induction hypothesis, we get $\beta \in \text{Mov}^0(Y/Z)$ such that $\mu_{p^*(\beta), \min}(H) > 0$, if $p : T \rightarrow Y$ is a Kawamata cover adapted to (Y, D_Y) , and $H := [p^*(\text{Ker}(dh)) \cap p^*(T(Y, D_Y))]^{\text{sat}}$, the saturation taking place in $p^*(T(Y, D_Y))$.

We may now, as in the preceding section, lift β to $\beta' \in \text{Mov}^0(X/Z)$ in such a way that $g_*(\beta') = \beta$, and $\beta'.E = 0$ for each component of the divisor $E' \subset X$, partially supported on the fibres of g , which supports Cokernel $(\mathcal{G}/\mathcal{H})$, as in the preceding subsection. We conclude just by the same arguments as above. \square

6.2. Relative differentials. We shall prove also, in this relative context, a relative version (used in §9 below) of the statement 2 in Theorem 1.1.

This is exactly the same statement as in [11], Proposition 3.10 and Theorem , up to the fact that we consider the full orbifold tensors, instead of their ‘round-ups’. The proof being the same (and even slightly simpler), we will be sketchy and refer to loc.cit for further details.

Let $f : (X, D) \rightarrow (Z, D_Z)$ be a surjective orbifold morphism of smooth complex projective orbifold pairs with connected fibres, (Z, D_Z) being the orbifold base of (f, D) . In this situation, and similarly to (but with modified notations) §4.4 and §4.3 above, we can construct a commutative diagram:

$$\begin{array}{ccccc} T & \xrightarrow{\rho} & Y & \xrightarrow{\pi} & X \\ & \searrow \sigma' & & & \downarrow f \\ W & \xrightarrow{q} & W' & \xrightarrow{p} & Z \end{array}$$

with the following properties:

1. $\pi : Y \rightarrow X$ (resp. $p : T \rightarrow Z$) are Kawamata covers adapted to D (resp. to D_Z).
 2. σ is connected (i.e. has connected fibres).
 3. q is generically finite, with W smooth.
 4. $p \circ q : W \rightarrow Z$ and $\pi \circ \rho : T \rightarrow X$ are Galois, T being normal.
- We also write: $\pi' := \pi \circ \rho$, and $p' := p \circ q \circ \sigma$.

Theorem 6.7. *Let $f : (X, D) \rightarrow (Z, D_Z)$ be a surjective orbifold morphism of smooth complex projective orbifold pairs with connected fibres, (Z, D_Z) being the orbifold base of (f, D) . Assume that the general orbifold fibre (X_z, D_z) of f is sRC. Then, in the above diagram:*

1. *One has, for any $m > 0$:*

$$\sigma_*(\otimes^m \pi'^*(\Omega^1(X, D))) = \otimes^m p^*(\Omega^1(Z, D_Z)).$$

2. *Moreover, if $L' \subset \otimes^m \pi'^*(\Omega^1(X, D))$ is a pseudo-effective line bundle on T , then $L' \subset \sigma^*(\otimes^m p^*(\Omega^1(Z, D_Z)))^{sat}$, the saturation being taken inside $\otimes^m \pi'^*(\Omega^1(X, D))$.*

Proof. Let us prove the statement 1 first. Let $U \subset Z - \text{Supp}(D_Z)$ be the dense Zariski open subset over which f , as well as its restriction to each component of D and of the nonempty intersections of the D_j 's is smooth. Over U , there is (after lifting to the Kawamata cover Y) a natural filtration of $\otimes^m(\pi^*(\Omega^1(X, D)))$ by subbundles with successive quotients $\otimes^A f^*(\Omega_U^1) \otimes^B Q$, where Q is the quotient bundle $[\pi^*(\Omega^1(X, D))/(f \circ \pi)^*(\Omega_U^1)]$, and $A \cup B = \{1, 2, \dots, m\}$ is a partition in two subsets A, B . The expression $\otimes^A E \otimes^B F$ denotes the set of tensors of the form: $t_1 \otimes \dots \otimes t_m$, with $t_k \in E$ (resp. $t_k \in F$), if $k \in A$ (resp. if $k \in B$).

Let $\alpha \in \text{Mov}^0(X/Z)$ be such that $\mu_{\pi^*(\alpha), \min}(\pi^*(T(X_z, D_z))) > 0$, for $z \in Z$ general. The existence of α is deduced from Theorem 6.6.

We thus get, for $m > 0$: $H^0(X_z, \otimes^A(\pi^*(\Omega^1(X_z, D_z)))) = \{0\}$

This implies that, over $V := (f \circ \pi)^{-1}(U) \subset Y$, one has:

$$H^0(V, \pi^*(\Omega^1(X, D))) = \pi^*(H^0(f^{-1}(U), f^*(\otimes^m(\Omega_U^1)))).$$

Thus over U , one has the equalities of sheaves, for any $\ell > 0$:

$$\sigma_*(\otimes^\ell \pi'^*(\Omega^1(X, D))) = p^*(\otimes^\ell \Omega_U^1) = \otimes^\ell p^*(\Omega^1(Z, D_Z)|_U).$$

We shall now show that these sections of $f^*(\otimes^m \Omega_U^1)$ over $f^{-1}(U)$, lifted to T by π , extend as sections of $(p \circ \sigma)^*(\otimes^m(\Omega^1(Z, D_Z)))$. This will imply the claim.

By Hartog's theorem, it is sufficient to show this over the complement in Z of the set of codimension 2 consisting of singular points of $\text{Supp}(D_Z)$. We can thus compute locally on Z , and assume that:

1. $\text{Supp}(D_Z)$ is given in local coordinates (z_1, \dots, z_p) , $p < n$ by the equation $z_1 = 0$,

2. that we have local coordinates (x_1, \dots, x_n) on X such that the support of D is given by the equation $x_1 = 0$, such that the map f is locally given by:

3. $f(x_1, \dots, x_n) := (z_1 = x_1^t, z_2 = x_2, \dots, z_p = x_p)$, and moreover that:

4. $D_Z = c'_1.(z_1 = 0)$, while $D = c_1.(x_1 = 0)$.

By the definition of the orbifold base of (f, D) , we shall, moreover, choose the chart of the x -coordinates centered at a point realising the minimum multiplicity of (f, D) over $z_1 = 0$.

This means that $c'_1 = 1 - \frac{a}{t.b}$, if $c_1 = 1 - \frac{a}{b}$, where t is as in 3. above.

We thus get: $f^*\left(\frac{dz_1}{z_1^{c'_1}}\right) = f^*(z_1^{1-c'_1} \cdot \frac{dz_1}{z_1}) = t.x_1^{1-c_1} \frac{dx_1}{x_1} = t \cdot \frac{dx_1}{x_1^{c_1}}$, and: $f^*\left(\frac{dz_i}{z_i}\right) = \left(\frac{dx_i}{x_i}\right)$, for $p \geq i \geq 2$. Let $c'_i = c_i = 0$, for $p \geq i \geq 2$. Then:

These equalities imply, symbolically, that, one has, for any multi-index $I := (i_1, \dots, i_m)$, the equality: $f^*\left(\otimes_{k=1}^{k=m} \left(\frac{dz_{i_k}}{z_{i_k}^{c'_{i_k}}}\right)\right) = \otimes_{k=1}^{k=m} \left(\frac{dx_{i_k}}{x_{i_k}^{c_{i_k}}}\right)$.

And so, symbolically: $f^*(\otimes^m(\Omega^1(Z, D_Z)))$ is, over $z_1 = 0$, saturated inside $\otimes^m(\Omega^1(X, D_X))$, outside of a divisor on X which is ‘partially supported on the fibres of f ’ (this is the same divisor F as in definition 2.12, and the few lines before it). This implies that: $f_*(\otimes^m(\Omega^1(X, D_X))) = \otimes^m(\Omega^1(Z, D_Z))$, in our situation of ‘sRC’ fibres, which concludes the proof (after lifting the ‘symbolic’ equalities above to T by π').

We now prove the statement 2. It is sufficient to show the claimed inclusion over the open set U defined above. But this is an immediate consequence of the filtration introduced above on $\otimes^m(\pi^*(\Omega^1(X, D)))$, since the minimal slopes of the successive terms of the associated graduation are strictly negative, except for the last one. Thus the projections of L' to the successive quotients, except for the last one, have to vanish, since L' is pseudo-effective. \square

The corollaries below will be used in the proof of Theorem 9.8.

Corollary 6.8. *The situation being as in Theorem 6.7, let $L' \in \text{Pic}(T)$ and $\ell > 0$ be such that $L' \subset \otimes^\ell \pi^*(\Omega^1(X, D))$. Assume also that $h^0(T, mL' + \pi^*(A)) \neq 0$ for infinitely many integers $m > 0$, A (resp. B) being ample divisors on X (resp. Z).*

There then exist an embedding $M' \subset \otimes^\ell p^(\Omega^1(Z, D_Z))^{sat}$ for some $M' \in \text{Pic}(W)$ such that*

$$H^0(T, mL' \pi^*(A)) \subset \oplus^r \sigma^*(H^0(W, mM' + p^*(kB))),$$

for any sufficiently large $m > 0$, the integers k, r being the ones defined in Lemma 6.10, depend only on A and B .

Proof. From the second claim of Theorem 6.7 we deduce that $L' \subset \sigma^*(\otimes^\ell(p^*(\Omega^1(Z, D_Z))))^{sat}$. Let $M' := \sigma_*(L')^{**}$: this is a reflexive rank-one coherent sheaf on W , by Lemma 6.9 below, applied to a generic fibre F of σ over $p^{-1}(U)$, and to the restriction of L' to F . Thus $M' \in \text{Pic}(W)$. Moreover, we may-and shall-assume L' to be saturated in $\otimes^\ell(\pi^*(\Omega^1(X, D)))$. We thus have: $L' = \sigma^*(M') + E$, where E is an effective divisor contained in the support of:

$$\sigma^*(\otimes^\ell(p^*(\Omega^1(Z, D_Z))))^{sat} / \sigma^*(\otimes^\ell(p^*(\Omega^1(Z, D_Z)))),$$

and thus ‘partially supported on the fibres of σ ’, after Proposition 2.13 (more precisely: E is contained in $\pi'^{-1}(F)$, where F is the divisor partially supported on the fibres of f described above this Proposition).

From Lemma 6.10 below, we deduce that for any sufficiently large $m > 0$, we have an injection $H^0(T, \sigma^*(mM') + A) \subset \oplus^r(\sigma^*H^0(W, mM' + p^*(kB)))$.

Finally, from Lemma 6.11, we get, for $m > 0$ sufficiently large:

$$H^0(T, mL' + A) = H^0(T, m \cdot \sigma^*(M') + A),$$

which establishes the claims. \square

Lemma 6.9. *Let F be a connected complex projective manifold, and $L \in \text{Pic}(F)$ a subbundle of a trivial bundle. Assume that L is pseudo-effective. Then L is trivial.*

Proof. The dual L^* is a quotient of a trivial bundle on F , and is thus generated by global sections. Assume that L is not trivial. Then $L^* = \mathcal{O}_F(D)$ for some nonzero effective divisor D on F .

Let A be ample on F . Since $h^0(F, mL + A) = h^0(F, -mD + A) = \{0\}$ if $m > 0$ is sufficiently large, L is not pseudo-effective. \square

Lemma 6.10. *Let $\sigma : T \rightarrow W$ be a proper map between compact connected normal projective varieties, T equipped with an effective line bundle A' and W with an ample line bundle B' .*

There then exist positive integers $k = k(A', B')$ and $r = r(A', B')$ such that, for any torsionfree coherent sheaf \mathcal{F} on W , one has an injection: $H^0(T, \sigma^(\mathcal{F}) \otimes A') \subset \oplus^r \sigma^*(H^0(W, \mathcal{F} \otimes \mathcal{O}_W(kB)))$.*

Proof. Notice that $\sigma_*(A')$ is a non-zero torsionfree sheaf which injects in its reflexive hull A'' , with dual $(A'')^*$. Now chose $k > 0, r > 0$ such that $(A'')^*(kB)$ is generated by r of its global sections, and dualise the surjection $\oplus^r \mathcal{O}_W \rightarrow (A'')^*(kB')$ so obtained. The claimed injection is then deduced from the natural sheaf injection $\sigma_*(A') \subset A'' \subset \oplus^r \mathcal{O}_W(kB')$, since $H^0(T, \sigma^*(\mathcal{F}) \otimes A') = \sigma^*(H^0(T, \mathcal{F} \otimes \sigma_*(A')))$. \square

Lemma 6.11. *Let $g : T \rightarrow W$ be a surjective connected morphism between complex projective manifolds. Let $E \subset T$ be an effective reduced divisor partially supported on the fibers of g .*

There exists an integer $k > 0$ such that, for any $N \in \text{Pic}(W)$, for any $s > k$, the natural injective map $H^0(T, N_k) \rightarrow H^0(T, N_s)$ is surjective, if $N_s := f^(N) + s \cdot E + A$.*

More precisely: let $n := \dim(T)$, and $d := \dim(T) - \dim(W) > 0$. Let A (resp. B) be a very ample line bundle on T (resp. W). Then one can chose $k := A^d \cdot E \cdot g^(B)^{n-d-1}$.*

Proof. This rests on the same considerations as in Proposition 3.6. Denote by S the smooth surface $A^{d-1} \cdot f^*(B)^{n-d-1} \subset X$, for generic members A, B (by abuse of notation) of the linear systems defined by the line bundles A, B . The reduced curve $S \cdot E := E' \subset S$ is partially supported on the fibers of $g_S : S \rightarrow B^{n-d-1}$, and the intersection number $S \cdot E' \cdot E''$ is thus strictly negative for each irreducible component E'' of E' .

Moreover, $g^*(N).E'''.S = 0$, and so, for any $s > k$:

$$N_s.E'''.S = s.E'.E'''.S + A.E'''.S \leq -s + A.E.S = -s + k < 0.$$

This then implies that $H^0(E, N_s) = 0$, and the surjectivity of the map: $H^0(S, N_k) \rightarrow H^0(S, N_s)$, by induction on $s > k$. Since the family of surfaces S covers X , we have the same statement for the sections over T (since the sections on T of any line bundle are determined by their restrictions to the S 's). \square

For the definition of the numerical dimension $\nu(X, L)$, we refer to §9. We shall now translate the preceding results using this notion.

Corollary 6.12. *In the above situation, and for L', M' defined as above, we have: $\nu(T, L') = \nu(W, M')$.*

Proof. We obviously have: $\nu(T, L') \geq \nu(W, M')$. In the other direction, we have from Corollary 6.8:

$\limsup_{s \rightarrow +\infty} \frac{h^0(T, mL' + \pi'^*(A))}{m^s} \leq r \cdot \limsup_{s \rightarrow +\infty} \frac{h^0(W, mM' + p'^*(kB))}{m^s}$, which is thus positive for the same values of s .

This establishes the claim \square

Let us notice a more general similar statement:

Corollary 6.13. *In the situation of Lemma 6.11, we have:*

$$\nu(W, N) = \nu(T, g^*(N)) = \nu(T, g^*(N) + \ell.E), \text{ for any } \ell \geq 0.$$

Proof. The first equality is shown in 9.2. For the second, use the inequalities, if $m.\ell \geq k$, and if A' is ample on T with $A' - (A + k.E)$ effective: $h^0(T, m.(g^*(N) + \ell.E) + A) = h^0(T, m.g^*(N) + k.E + A) \leq h^0(T, m.g^*(N) + A')$. The conclusion then follows from the definition of the numerical dimension. \square

7. ORBIFOLD SLOPE RATIONAL QUOTIENT

We shall show here the existence of a ‘rational quotient’ ([6], see also [25] under the name ‘MRC-fibration’) in the orbifold context.

Theorem 7.1. *Let (X, D) be smooth, complex projective and connected. There exists (on some suitable birational model) an orbifold morphism which is a fibration $\rho : (X, D) \rightarrow (R, D_R)$ onto its (smooth) orbifold base (R, D_R) which has the following two properties:*

1. *Its smooth orbifold fibres (X_r, D_r) are sRC ($X_r := \rho^{-1}(r)$, $r \in R$).*
2. *$K_R + D_R$ is pseudo-effective.*

Of course, $R=X$ (resp. R is a point) if and only if $(K_X + D)$ is pseudo-effective (resp. if and only if (X, D) is sRC).

This fibration is unique, up to orbifold birational equivalence. It is, moreover, characterised by any one of the following two properties:

3. *$\dim(X) - \dim(Z)$ is maximal among the fibrations $f : X \dashrightarrow Z$ such that (X_z, D_z) is sRC for generic $z \in Z$.*

4. $\dim(Z)$ is maximal such that $K_Z + D_Z$ is pseudo-effective, among all fibrations $f : X \dashrightarrow Z$, (Z, D_Z) being the orbifold base (on any ‘neat’ orbifold birational model, here and also in 3. above)

The proof will be obtained below by combining Theorem 1.1 with the following factorisation criterion, and its Corollary 7.3:

Proposition 7.2. *Let (X, D) be as above, together with two orbifold morphisms which are fibrations over their orbifold bases: $f : (X, D) \rightarrow (R, D_R)$ and $g : (X, D) \rightarrow (Z, D_Z)$. Assume that:*

1. $K_R + D_R$ is pseudo-effective.
2. The generic orbifold fibre (X_z, D_z) of g is sRC.

Then: there exists a rational map $h : Z \rightarrow R$ such that $h \circ g = f$.

Proof. Denote by $R_z := f(X_z) \subset R$ the image of a general fibre X_z of g by f , and assume by contradiction that $\dim(R_z) > 0$, since the claims amounts to prove that $\dim(R_z) = 0$. The commutative diagram:

$$\begin{array}{ccccc} (X_z, D_z) & \xrightarrow{j_z} & (X, D) & \xrightarrow{g} & Z \\ \downarrow f_z & & \downarrow f & & \\ R_z & \xrightarrow{r_z} & (R, D_R) & & \end{array}$$

induces, for $m > 0$ sufficiently divisible, a commutative diagram, the maps r_z^* and f_z^* being defined on R_z^{reg} only:

$$\begin{array}{ccc} (K_R + D_R)^{\otimes m} & \xrightarrow{r_z^*} & K_{R_z}^{\otimes m} \\ \downarrow f^* & & \downarrow f_z^* \\ \pi^*(\Omega^1(X + D))^{\otimes m} & \xrightarrow{j_z^*} & \pi^*(\Omega^1(X_z + D_z))^{\otimes m} \end{array}$$

The map $(f_z^* \circ r_z^*)$ is generically injective, and of generic rank 1, because $\dim(R_z) > 0$. Thus $(j_z^* \circ f^*)$ also has generic rank 1. By assumption, we have, for a certain movable class α on X such that $g_*(\alpha) = 0$: $\mu_{\alpha, \max}(\Omega^1(X_z, D_z)) < 0$. Since $K_R + D_R$ is pseudo-effective, so is $(K_R + D_R)_z := (K_R + D_R)|_{X_z}$, and so: $\alpha \cdot (f^*(K_R + D_R)_z) \geq 0$, which contradicts the injection $\pi_z^*((K_R + D_R)_z^{\otimes m}) \subset \pi_z^*(\Omega^1(X_z, D_z)^{\otimes m})$, if $\pi_z : Y_z \rightarrow X_z$ is the restriction to X_z of a Kawamata cover adapted to (X, D) , which is a Kawamata cover adapted to (X_z, D_z) . \square

Corollary 7.3. *Let (X, D) be smooth projective, together with an orbifold morphism $g : (X, D) \rightarrow (Z, D_Z)$ which is a fibration with (Z, D_Z) its orbifold base. Assume that both (Z, D_Z) and the general orbifold fibre (X_z, D_z) of g are sRC. Then so is (X, D) .*

Proof. Let $\rho : (X, D) \rightarrow (R, D_R)$ be (on some suitable birational model) an orbifold morphism which is a fibration onto its orbifold base, with $K_R + D_R$ pseudo-effective. We want to show that $\dim(R) = 0$. By

Proposition 7.2, we get a factorisation $h : Z \rightarrow R$ such that $\rho = h \circ g$. But now, g being an orbifold morphism, (R, D_R) is also the orbifold base of $h : (Z, D_Z) \rightarrow R$. Because we assumed (Z, D_Z) to be *sRC*, we get: $\dim(R) = 0$ as claimed. \square

Proof. (of Theorem 7.1) We proceed by induction on $n := \dim(X)$. When $n = 1$, everything is clear, since (X, D) is either *sRC* or has pseudo-effective canonical bundle $K_X + D$. We thus assume that $n \geq 2$, and that the conclusion of Theorem 7.1 holds whenever $n' < n$.

Existence of ρ : If $K_X + D$ is pseudo-effective, $\rho := id_X$. Otherwise, there exists (on a suitable birational model of our initial (X, D)) a fibration $g : (X, D) \rightarrow (Z, D_Z)$ as in the proposition 7.2 with *sRC* orbifold fibres, and $\dim(Z) < n$. The conclusion of the theorem 7.1 thus applies to (Z, D_Z) . By taking further birational models, we have a ‘rational quotient’ $\rho' : (Z, D_Z) \rightarrow (R, D_R)$. The Corollary 7.3 now shows that the orbifold fibres of $\rho := \rho' \circ f : (X, D) \rightarrow (R, D_R)$ are *sRC*. Because $f : (X, D) \rightarrow (Z, D_Z)$ was an orbifold morphism, the orbifold base of $\rho : (X, D) \rightarrow R$ is (R, D_R) . And ρ thus possesses the two characteristic properties of a ‘rational quotient’.

Uniqueness of ρ : Let $\rho : (X, D) \rightarrow (R, D_R)$ and $\rho' : (X, D) \rightarrow (R', D_{R'})$ be two fibrations having the two characteristic properties stated in Theorem 7.1. By Proposition 7.2, we have factorisations $h : R \rightarrow R'$ (resp. $h' : R' \rightarrow R$) such that $\rho' = h \circ \rho$ (resp. such that $\rho = \rho' \circ h'$). Thus $R = R', \rho = \rho'$.

Uniqueness of (R, D_R) up to birational equivalence: This is a general property of ‘neat’ birational models of fibrations. See Section 2.5.

Let us now check that ρ is characterised by any of the properties 3. or 4. in the statement of Theorem 7.1. Let $\rho : (X, D) \rightarrow (Z, D_Z)$ be the ‘sRC quotient’, and let $g : (X, D) \rightarrow (Y, D_Y)$ be another (neat) fibration.

Assume first that the generic fibres (X_y, D_y) are *sRC* (resp. that $K_Y + D_Y$ is pseudo-effective). Then, from Proposition 7.2, we deduce the existence of some $h : Y \dashrightarrow Z$ such that $\rho = h \circ g$ (resp. $h : Z \rightarrow Y$ such that $g = h \circ \rho$). We thus have the maximality properties of statements 3 (resp. 4) if and only if $g = \rho$ \square

We shall need a relative version of this ‘sRC’-quotient in the next section. We abuse notation in the sequel, by still writing (X, D) for any suitable orbifold birational model of our initial (X, D) , in order to simplify notations. Also a factorisation of a fibration $f = g \circ r$ of $f : X \dashrightarrow Y$ will be a pair (r, g) of fibrations $r : X \dashrightarrow Z$, $g : Z \rightarrow Y$ such that $f = g \circ r$. We shall always chose (after suitable orbifold modifications) $r : (X, D) \rightarrow (Z, D_Z)$ to be a neat orbifold morphism to its orbifold base, and similarly for h . Moreover, $\text{Ker}(dr), \text{Ker}(dg), \text{Ker}(f)$ determine D, D_Z, D -orbifold-foliations on X, Z, X respectively (their

construction is recalled just before Theorem 8.1 below). These orbifold foliations will be denoted by $\mathcal{R}_D, \mathcal{G}, \mathcal{F}_D$ respectively:

Theorem 7.4. *Let (X, D) be a smooth orbifold pair, and $f : X \rightarrow Y$ be a fibration. There exists then a (birationally) unique factorisation $f = g \circ \rho_f$, with $\rho_f : (X, D) \rightarrow (Z, D_Z)$, and $g : Z \rightarrow Y$, such that, for $y \in Y$ general: $\rho_f|_{X_y} : (X_y, D_y) \rightarrow (Z_y, (D_Z)_y)$ is the ‘sRC quotient’ of (X_y, D_y) , the orbifold fibre of f over y , $(Z_y, (D_Z)_y)$ being the orbifold fibre of $g : (Z, D_Z) \rightarrow Y$.*

Moreover, the D_Z -foliation \mathcal{G} defined by $\text{Ker}(dg)$ has a pseudo-effective canonical bundle.

The factorisation $f = g \circ \rho_f$ is characterised, among all factorisations $f = g \circ r$, by any one of the following two properties:

2. The general fibres (X_z, D_z) of r are ‘sRC’, and $\dim(Z)$ is minimal for these properties.

3. The D_Z -foliation \mathcal{G} determined by $\text{Ker}(dg)$ has pseudo-effective canonical bundle, and $\dim(Z)$ is maximal for these properties.

The factorisation $f = g \circ \rho_f$ is called ‘the slope rational quotient of f ’.

Proof. (of Theorem 7.4) The uniqueness is clear. To show the existence, we proceed by induction on $d := \dim(X) - \dim(Y)$, the assertion being obvious when $d = 0$. If $K_{\mathcal{F}}$ is pseudo-effective, $Y = X$ satisfies the assertions. Otherwise, by Corollary 6.2, there exists a factorisation $f = g \circ r$, with $\dim(Z) < \dim(X)$ such that $r : (X, D) \rightarrow (Z, D_Z)$ has ‘sRC’ general fibres (X_z, D_z) . This Corollary can, indeed, be applied in the relative setting, by Remark 6.3. By induction hypothesis, we thus have a ‘slope RC quotient’ factorisation $g = h \circ \rho_g$ for g , with $\rho_g : (Z, D_Z) \rightarrow (T, D_T)$ having ‘sRC’ general fibres, and $h : (T, D_T) \rightarrow Y$ such that $K_{\mathcal{H}}$ is pseudo-effective, where \mathcal{H} is the orbifold on (T, D_T) associated to $\text{Ker}(dh)$. We obtain in this way By Corollary 7.3, the fibres of $\rho_f := \rho_g \circ r : (X, D) \rightarrow (T, D_T)$ are ‘sRC’, and (T, D_T) is the orbifold base of this fibration, since both ρ_g and r have been chosen to be orbifold morphisms. The factorisation $f = h \circ \rho_f$ thus fulfills the two conditions for being a relative ‘sRC quotient of f ’, since for any orbifold foliation \mathcal{H} on (T, D_T) over Y , $K_{\mathcal{H}}$ is pseudo-effective if and only if so is its restriction to any fibre T_y (by [15], Theorem 6.2). \square

Remark 7.5. *In general, the orbifold rational quotient $\rho : (X, D) \dashrightarrow R$ is not ‘almost holomorphic’ (see again the example 6.17, p. 859, of [9]: $(\mathbb{P}^2, L_1 + L_2)$ if $L_i, i = 1, 2$ are two distinct lines). However, ρ is almost holomorphic if (X, D) is klt, by [9], Theorem 9.19, p. 896 (the proof applies to our slightly more general situation).*

8. ORBIFOLD FOLIATIONS OF POSITIVE SLOPE.

Let (X, D) be a smooth projective orbifold, $\pi : Y \rightarrow X$ a Kawamata cover adapted to (X, D) , and $\mathcal{F}_D \subset \pi^*(T(X, D))$ a foliation on

(X, D) . We say that \mathcal{F}_D is a D -foliation. If $f : X \dashrightarrow Z$ is a rational dominant fibration, it defines a foliation $\mathcal{F} := \text{Ker}(df)$ on X , and a D -foliation $\mathcal{F}_D := \pi^*(\mathcal{F}) \cap \pi^*(T(X, D)) \subset \pi^*(TX)$. Conversely, if $\mathcal{F}_D \subset \pi^*(T(X, D))$ is a D -foliation (see [15] for this notion), it defines a foliation \mathcal{F} on X characterised by the equality: $\mathcal{F}_D^{\text{sat}} = \pi^*(\mathcal{F})$, where $\mathcal{F}_D^{\text{sat}}$ is the saturation in $\pi^*(TX)$ of \mathcal{F}_D . And \mathcal{F} is algebraic means that its leaves are algebraic, or equivalently, that $\mathcal{F} = \text{Ker}(df)$ for some rational dominant fibration $f : X \dashrightarrow Z$. In this case,

Theorem 8.1. *Assume that $\mathcal{F}_D \subset \pi^*(T(X, D))$ is a D -foliation, and that $\mu_{\alpha', \min}(\mathcal{F}_D) > 0$ for some movable class α on X , where $\alpha' := \pi^*(\alpha)$ is movable on Y . Then:*

1. \mathcal{F} is algebraic, let $f : X \dashrightarrow Z$, be such that $\mathcal{F} = \text{Ker}(df)$.
2. On any ‘neat’ orbifold birational model $f' : (X', D') \rightarrow Z'$ of f , the generic orbifold fibre (X'_z, D'_z) of f' is sRC.

Conversely, if (f, D) possesses the above property 2, the D -foliation \mathcal{F}_D associated to it¹⁵ has $\mu_{\alpha', \min}(\mathcal{F}_D) > 0$ for some α movable on X , and for any Kawamata cover π adapted to (X, D) .

Proof. The proof is a direct adaptation to the orbifold context of the proof given in [15] when $D = 0$. From [15], we know that the foliation \mathcal{F} on X defined by the saturation of \mathcal{F}_D in $\pi^*(TX)$ is algebraic. Let $f : (X, D) \rightarrow Y$ be a neat orbifold birational model of the rational fibration $f : (X, D) \rightarrow Y$ defined by \mathcal{F}_D . We know from [15] that its generic orbifold fibres (X_y, D_y) have a canonical bundle $K_{X_y} + D_y$ which is not pseudo-effective. Let $\rho_f : (X, D) \rightarrow (Z, D_Z)$ be its ‘sRC quotient’, with the factorisation $f = g \circ \rho_f$, and $g : (Z, D_Z) \rightarrow Y$. We thus know that $K_{\mathcal{G}}$ is pseudo-effective, if $\mathcal{G} \subset p^*(T(Z, D_Z))$ is the D_Z -foliation defined by the foliation $\mathcal{G}_Z := \text{Ker}(dg) \subset TZ$ on Z . We have¹⁶ a natural derivative map: $\pi^*(d\rho_f) : \pi^*(\mathcal{F}_D) \rightarrow (\rho_f)^*(q^*(\mathcal{G}))$ which is generically surjective (note that this map is, generically on X , nothing but $df : \mathcal{F} \rightarrow \rho_f^*(\mathcal{G}_Z)$).

Assume that $\dim(Z) > \dim(Y)$, or equivalently, that $\mathcal{G} \neq 0$.

We thus have: $0 < \mu_{\pi^*(\alpha), \min}(\pi^*(T(X, D))) \leq \mu_{\pi^*(\alpha), \min}(\rho_f^*(q^*(\mathcal{G})))$. But this contradicts the fact that $K_{\mathcal{G}}$ is pseudo-effective. \square

9. BIRATIONAL STABILITY OF THE ORBIFOLD COTANGENT BUNDLE

9.1. Numerical dimension.

Definition 9.1. *Define, if A sufficiently ample, and $L \in \text{Pic}(X)$, the ‘numerical dimension’ of L to be:*

$$\nu(X, L) := \max\{k \in \mathbb{Z} \mid \overline{\lim}_{m \rightarrow 0} \left(\frac{h^0(X, mL + A)}{m^k} \right) > 0\}.$$

¹⁵By the construction recalled before the statement of Theorem 8.1.

¹⁶On a suitable finite cover of X' dominating a Kawamata cover $q : Z' \rightarrow Z$ adapted to (Z, D_Z) still denoted X' , see the constructions made in §4.4.

Recall some easy properties:

1 $\kappa(X, L) \leq \nu(X, L) \in \{-\infty, 0, 1, \dots, n\}$.

2 $\nu(X, L + P) \geq \nu(X, L)$ if $P \in \text{Pic}(X)$ is pseff.

In general, there is no further relationship between $\nu(X, L)$ and $\kappa(X, L)$, except in the following extremal case:

3 $\kappa(X, L) = n$ if $\nu(X, L) = n$.

We shall need the following easy property:

Lemma 9.2. *Let $\pi : T \rightarrow X$ be a proper morphism between two normal connected complex projective varieties, and let $L \in \text{Pic}(X)$, together with a sufficiently ample line bundle A on X . Then $\nu(X, L) = \nu(T, \pi^*(L)) = \max\{s \in \mathbb{Z} \mid \limsup_{s \rightarrow +\infty} \frac{h^0(T, \pi^*(mL + A))}{m^s} > 0\}$.*

Proof. We only prove the first equality, which obviously implies the second one, which we now prove. Let thus A' be ample on T . From Lemma 6.10, we have a natural injection $\pi_*(A') \subset \oplus^r \mathcal{O}_X(kA)$, for some integers $k > 0, r > 0$. This implies the inequality: $h^0(T, \pi^*(mL) + A') \leq r \cdot h^0(X, mL + kA)$, which easily implies that $\nu(T, \pi^*(L)) \leq \nu(X, L)$, the reverse implication being obvious. \square

A central result is the following:

Theorem: ([BDPP], [Nak]) L pseff iff $\nu(X, L) \geq 0$.

9.2. Maximal numerical dimension of a coherent sheaf. The following notion was introduced in [11]:

Definition 9.3. *Let X be a connected normal complex projective variety, and \mathcal{F} a torsion free coherent sheaf on X . Define: $\nu^+(X, \mathcal{F}) := \max\{\nu(X, L) \mid L \in \text{Pic}(X), L \subset \otimes^m \mathcal{F}, m > 0\}$.*

We shall need the following elementary property:

Lemma 9.4. *Let $\pi : T \rightarrow X$ be a generically finite Galois map between connected complex projective manifolds. Let \mathcal{F}' be a torsionfree coherent sheaf on T , equipped with an equivariant action of the group $\text{Gal}(T/X) := G$.*

Define:

$$\nu^+(X, \mathcal{F}', G) := \max\{\nu(X, L) \mid L \in \text{Pic}(X), \pi^*(L) \subset \otimes^m(\mathcal{F}'), m > 0\}$$

$$\text{Then: } \nu^+(X, \mathcal{F}', G) = \nu^+(T, \mathcal{F}').$$

Proof. The inequality $\nu^+(X, \mathcal{F}', G) \leq \nu^+(T, \mathcal{F}')$ is obvious. In the reverse direction, let $L' \in \text{Pic}(T), L' \subset \otimes^m \mathcal{F}'$. We assume that L' is pseudo-effective, since otherwise the claim is clear.

Consider $L'' := \otimes_{g \in G} g^*(L') \subset \otimes^{mN} \mathcal{F}'$, if $N = \text{Card}(G)$. Since L'' is G -invariant, there exists $L \in \text{Pic}(X)$ such that $L'' = \pi^*(L) + E^+ - E^-$, where E^+, E^- are effective divisors supported on the exceptional locus of π , and without common components. By Hartog's theorem, we thus have: $\nu(T, L'') \leq \nu(T, \pi^*(L) + E^+) = \nu(X, L)$.

On the other hand, we also have: $\nu(T, L'') \geq \nu(T, L')$, since each of the $g^*(L')$ is effective. This implies that $\nu(T, L') \leq \nu(X, L)$, as claimed. \square

We shall apply this notion to the orbifold cotangent and canonical bundles.

Definition 9.5. *Define:* $\nu(X, D) := \nu(X, K_X + D) \geq \kappa(K_X + D)$.

$$\nu^+(X, D) := \max\{\nu(X, L) \mid L \in \text{Pic}(X), \pi^*(L) \subset \otimes^m(\pi^*(\Omega^1(X, D))), m > 0\}$$

It is independent on π , since $\nu(Y, \pi^(L)) = \nu(X, L)$, by Lemma 9.2.*

From Lemma 9.4, we also see that: $\nu^+(X, D) = \nu^+(T, \pi'^(\Omega^1(X, D)))$, for any Kawamata cover π adapted to (X, D) , and any $\pi' = \pi \circ \rho$ as in Theorem 6.7.*

The same notions have been introduced in [11], using the (essentially equivalent) sheaves $[S^m](X, D)$, and also in [10], but using κ instead of ν there. Using κ however presently leads to conjectures, rather than theorems, as below.

Obviously: $\nu^+(X, D) \geq \nu(X, D)$. We shall, in the next two sections, revert this inequality.

9.3. $K_X + D$ pseudoeffective: $\nu^+(X, D) = \nu(X, D)$.

We just recall here:

Theorem 9.6. ([15], Theorem 7.3) *Let (X, D) be a smooth (projective, connected, complex) orbifold with $K_X + D$ pseudoeffective. Then $\nu^+(X, D) = \nu(X, D)$*

An important special case is:

Corollary 9.7. ([15], Theorem 7.7) *Let (X, D) be a smooth (connected complex projective) orbifold pair. If $\nu(X, D) = 0$, then $\nu^+(X, D) = 0$.*

If $K_X + D \equiv 0$, then: $-L$ is pseudo-effective, if $L \in \text{Pic}(X)$ is such that $\pi^(L) \subset \otimes^m(\pi^*(\Omega^1(X, D)))$ for some $m > 0$. In particular: $L \cong \mathcal{O}_X$ if $h^0(Y, \pi^*(L)) \neq 0$, and: $\kappa(Y, \pi^*(L)) \leq 0$.*

9.4. General case: $\nu^+(X, D) = \nu(R, D_R)$.

Theorem 9.8. *Let (X, D) be a smooth (projective, connected, complex) orbifold pair. Let $r : (X, D) \rightarrow (R, D_R)$ be its ‘slope Rational Quotient’ (on a suitable orbifold birational, strictly neat, model). Then $\nu^+(X, D) = \nu(R, D_R)$*

Proof. The inequality $\nu^+(X, D) \geq \nu(R, D_R)$ is indeed obvious.

The reverse inequality is an immediate consequence of the corollary 6.12, combined with the equalities $\nu^+(X, D) = \nu^+(T, \pi'^*(\Omega^1(X, D)))$ and $\nu^+(Z, D_Z) = \nu^+(W, p^*(\Omega^1(Z, D_Z)))$ observed in Definition 9.5, the last one applied to $(Z, D_Z) = (R, D_R)$. The notations T, W , are those of the diagram introduced before the statement of Theorem 6.7. \square

In particular, let us stress that:

Corollary 9.9. *Let (X, D) be a smooth (projective, connected, complex) orbifold pair.*

Then: (X, D) is sRC if and only if $\nu^+(X, D) = -\infty$.

We recover in a more natural way, and in a more general context than in [15] the criterion for being of Log-general type:

Corollary 9.10. ([15], Theorem 7.6) *Let (X, D) be a smooth (projective, connected, complex) orbifold pair.*

Assume that $\nu^+(X, D) = n$. Then: $\kappa(X, D) = n$.

Proof. Since $n = \nu^+(X, D) = \nu(R, D_R) \leq \dim(R) \leq n$, we have $\dim(R) = n$, and $X = R$, which means that $(X, D) = (R, D_R)$, and so $\nu(X, D) = \nu(R, D_R) = n$, as asserted. \square

10. FANO ORBIFOLDS.

We prove here Theorem 10.1 as an easy application of Theorem 2.11 in [14].

Recall the statement to be proved:

Theorem 10.1. *Let (X, D) be a smooth orbifold pair which is klt¹⁷, and Fano (ie: $-(K_X + D)$ is ample on X). Then (X, D) is slope rationally connected.*

Proof. Let $g : (X', D') \rightarrow (X, D)$ be a birational morphism, with (X', D') smooth such that D' is the strict transform of D in X' , together with a fibration $f : X' \rightarrow Z$ on a smooth projective variety Z . We assume $f : (X', D') \rightarrow Z$ to be ‘neat’ and its orbifold base (Z, D_Z) to be smooth. Because we assumed (X, D) to be klt, we can write: $g^*(K_X + D) = K_{X'} + D' + \Delta'$, where Δ' is supported on the exceptional locus of g , and where $(X', (D' + \Delta'))$ is again klt. Let H be ample on Z . Since $A := -(K_X + D)$ is ample, we can write $g^*(A) = \vartheta.E' + A' + \varepsilon.f^*(H)$, where $\vartheta > 0, \varepsilon > 0$ are chosen to be sufficiently small, and E' is the reduced support of the exceptional divisor of g . We can thus write: $0 = g^*(K_X + D + A) = K_{X'} + D' + (\Delta' + \vartheta.E') + A' + \varepsilon.f^*(H)$. Since A', H are ample and $(\Delta' + \vartheta.E')$ is supported on E' and $(X', (\Delta' + \vartheta.E'))$ is klt for $0 < \vartheta$ sufficiently small, we can chose \mathbb{Q} -divisors linearly equivalent to A' and H in such a way that:

1. $D'' := D' + E' + A' + \varepsilon.f^*(H)$ has snc support and is such that: (X', D'') is lc.
2. A' is f -horizontal (for example: its support is irreducible)
3. $D_Z + H + f(D'^{vert})$ has snc support, where $(\bullet)^{hor}$ and $(\bullet)^{vert}$ stand for the f -horizontal and f -vertical parts of a divisor (\bullet) on X' .

¹⁷This means that all coefficients of the components of D are less than 1, strictly.

We then have: $K_{X'} + D'' = E''$, where E'' is supported on E' and (X', E'') is klt. Moreover: $g : (X', D'') \rightarrow (X, D)$ is an orbifold morphism (because we have equipped all the components of E' with multiplicities $+\infty$, which was the reason to replace $(\Delta' + \vartheta.E')$ by E')

Let now (Z, D_Z) be the orbifold base of $f : (X', D'') \rightarrow Z$, which is also ‘neat’. We have, for its orbifold base (Z, D'_Z) : $D'_Z = D_Z + \varepsilon.H$, by our generic choice of the \mathbb{Q} -divisor H on Z .

Now, by [14], Theorem 2.11: $K_{X'/Z} + (D'')^{\text{vert}} - D(f, 0)$ is pseudo-effective. Here $D(f, 0)$ is an effective f -vertical divisor for the definition of which we refer to loc.cit. In particular, $K_{X'/Z} + D'' - f^*(D_Z + \varepsilon.H) = E'' - f^*(K_Z + D_Z + \varepsilon.H)$ is pseudo-effective. Let $C \subset X$ be the complete intersection of ample divisors avoiding $g(E')$, and C' its inverse image in X' ; then $g_*(C').(K_Z + D_Z) \leq -g_*(C').(\varepsilon.H < 0$, since $E'.C' = 0$. This implies that $-(K_Z + D_Z)$ is not pseudo-effective, since $[g_*(C')] \in \text{Mov}(Z)$. \square

One can remove the klt assumption in Theorem 10.1 under the vanishing of Log- p -forms:

Theorem 10.2. *Let (X, D) be a smooth orbifold pair with $-(K_X + D)$ ample. Assume that:*

1. *D is reduced¹⁸*
 2. *$H^0(X, \Omega_X^q(\text{Log}(D))) = 0$, for all $q > 0$.*
- Then:*
3. *$H^0(X, \otimes^m(\Omega_X^1(\text{Log}(D)))) = 0$, for all $m > 0$.*
 4. *X is rationally connected.*

Proof. We just need to apply Corollary 2.10 of [15]. Indeed, $\text{Pic}(X)$ has no torsion, since $\pi_1(X) = \{1\}$ by [5], X being rationally connected by Corollary 10.3 above. \square

When (X, D) is smooth and Fano, but not necessarily klt, one still gets:

Theorem 10.3. *Let (X, D) be a smooth orbifold which is Fano, and let (Z, D_Z) be the orbifold base of any neat orbifold birational model of any rational dominant fibration $f : X \dashrightarrow Z$. Then:*

1. *If $(K_Z + D_Z)$ is pseudo-effective, $\nu(Z, K_Z + D_Z) = 0$, where $\nu(Z, L)$ is the ‘numerical dimension’ of the \mathbb{Q} -divisor L .*
2. *X is rationally connected.*

Proof. For Claim 1, argue as in the proof of Theorem 10.1, writing $0 = g^*(K_X + D + A) = K_{X'} + D' + (\Delta') + A'$, where this time A' is a \mathbb{Q} -divisor linear equivalent to $g^*(A)$ choosen sufficiently generic, so that the support of $D'' := D' + E' + A'$ is snc. We then get, for

¹⁸This is actually not necessary, the proof still works in the general case. This just simplifies the statements.

the orbifold base (Z, D_Z) of $f : (X', D'') \rightarrow Z$, from [14], 2.11 again: $K_{X'/Z} + D'' - f^*(D_Z) := P$ is pseudo-effective. On the other hand, this \mathbb{Q} -divisor P is linearly equivalent to: $E'' - f^*(K_Z + D_Z)$, with $E'' := E' - \Delta'$, which is effective and g -exceptional. If $K_Z + D_Z$ is pseudo-effective, we thus get: $f^*(K_Z + D_Z) = E'' - P$, and thus:

$$\nu(Z, K_Z + D_Z) \leq \nu(X', f^*(K_Z + D_Z)) \leq \nu(X', E'') = 0.$$

Since $K_Z + D_Z$ is pseudo-effective, we have: $\nu(Z, K_Z + D_Z) \geq 0$, and Claim 1.

For Claim 2, it suffices to note that K_Z is not pseudoeffective if so is $K_Z + D_Z$. \square

Remark 10.4. *Claim 2 was already observed in [31]. Using Theorem 2.11 of [14] instead of Log-subadjunction techniques simplifies considerably the proof. Kawamata's base-point freeness, permits to easily extend the above results to the case when $-(K_X + D)$ is nef and big.*

From Theorem 10.3, one immediately gets:

Corollary 10.5. *Let (X, D) be a smooth Fano orbifold. Then $\nu^+(X, D) \in \{-\infty, 0\}$.*

Let us illustrate the preceding results by the following simple:

Example: Let $(X, D) := (\mathbb{P}_n, D_k = H_1 + \dots + H_k)$, H_j hyperplanes in general position, $2 \leq k \leq n$: it is Fano, but 'purely logarithmic' (i.e: the coefficients of the components of $D = D_k$ are all equal to 1).

Its slope rational quotient is the linear projection $\pi : \mathbb{P}_n \rightarrow \mathbb{P}_{k-1}$ centered at the intersection of the H_j 's. The orbifold base of this projection is (\mathbb{P}_{k-1}, D'_k) , with $D'_k = \pi(D_k)$, which has trivial Log-cotangent bundle.

We thus have: $\nu^+(X, D) = 0$, $h^0(X, \Omega_X^k(\text{Log}(D))) = 1$, and: $h^0(X, \Omega_X^q(\text{Log}(D))) = 0$, for all $0 < q \neq k$.

11. ORBIFOLD RATIONAL CURVES: CONJECTURES

In this last section, we introduce the notions of orbifold rational curves, orbifold uniruledness, and rational connectedness, extracted from [10] and [11], to which we refer for more details and justifications for the notions introduced here. We then state the conjectures stating the equivalence between 'slope rational connectedness' and orbifold rational connectedness, again as in loc. cit.. A much weaker property will then be shown in the next section 12.

In the present section, (X, D) will be a smooth, connected, and complex-projective orbifold pair, with $D = \sum_{j \in J} c_j \cdot D_j$, the 'coefficients' $c_j := (1 - \frac{1}{m_j}) \in]0, 1]$ being rational numbers, with the 'multiplicities' $m_j := \frac{a_j}{b_j} = (1 - c_j)^{-1} \in \mathbb{Q}$, with $0 < a_j \leq b_j$ being coprime integers. We also consider another smooth projective connected orbifold pair (C, D_C) , in which C is a curve (so that $D_C := \sum_{k \in K} c'_k \cdot p_k$,

the p'_k s being distinct points of C , and $c'_k \in \mathbb{Q} \cap]0, 1], \forall k \in K$). Write again: $c'_k := (1 - \frac{1}{m'_k})$ for the corresponding multiplicities (on C).

Definition 11.1. ([11], Definition 9) *Let $f : C \rightarrow X$ be a morphism. We say that $f : (C, D_C) \rightarrow (X, D)$ is an orbifold morphism if:*

1. f is birational from C to $f(C)$.
2. $f(C) \subsetneq \text{Supp}(D)$.
3. For each $a \in C$ and each $j \in J$ such that $f(a) \in D_j$, let $t_{a,j}$ be the order of contact of $f(C)$ and D_j at $f(a)$, that is: $f^*(D_j) = t_{a,j} \cdot \{a\} + \dots$.

An immediate computation shows that $f : (C, D_C) \rightarrow (X, D)$ is an orbifold morphism if and only if: $df : TC \rightarrow f^*(TX)$ induces an injection of sheaves $g^*(df) : g^*(T(C, D_C)) \rightarrow g^*(f^*(T(X, D)))$, for any finite map $g : C' \rightarrow C$ such that C' dominates a Kawamata cover of C adapted to D_C , and such that $f \circ g : C' \rightarrow X$ factorises through some Kawamata cover $\pi : Y \rightarrow X$ adapted to D .

This former definition is thus justified by this property. In the texts [10], [11], a similar justification was given in terms of (slightly different) variants of the orbifold tangent bundle of (X, D) (and for morphisms from higher-dimensional C' s, not necessarily curves).

Notice also that, for given X, D, C there is smallest orbifold divisor D_C for which $f : (C, D_C) \rightarrow (X, D)$ is an orbifold morphism. For each $a \in C$, it attributes to the point a the coefficient $c_{(f,a)} = (1 - \frac{1}{m_{(f,a)}})$, with $m_{(f,a)} := \inf\{t_{a,j} \cdot m_j\}$, where $j \in J$ is such that $f(a) \in D_j$. Unless explicitly said, we shall, in the sequel, although consider as D_C this minimal orbifold structure making f an orbifold morphism.

Definition 11.2. ([11], Definition 9 and Definition 10) *Let $f : (C, D_C) \rightarrow (X, D)$ be an orbifold morphism, as above. Then (C, D_C) is said to be D -rational curve if $\deg(K_C + D_C) < 0$. This clearly implies that $C \cong \mathbb{P}^1$.*

Definition 11.3. ([11], Definition) *We say that (X, D) is uniruled (resp. Rationally connected) if there exists an irreducible orbifold rational curve going through any generic point (resp. any generic pair of points) of X . We say that (X, D) is ‘weakly uniruled’ if, through the generic point of X , there exists an irreducible rational curve C with $(K_X + D) \cdot C < 0$.*

Of course, there are many stronger variants of this notion. See [9] for some of them.

We can now state the main conjecture concerning orbifold rational curves (see [11], for similar conjectures, but related to κ , rather than ν):

Conjecture 11.4. *Let (X, D) be a smooth projective¹⁹ orbifold pair.*

¹⁹Or Compact Kähler, or in the class C.

1. (X, D) is uniruled if and only if $K_X + D$ is not pseudo-effective.
- 1'. (X, D) is uniruled if and only if it is weakly uniruled²⁰.
2. (X, D) is rationally connected if and only if it is ‘slope rationally connected’.

Remark 11.5. *One has the following easy implications, for a smooth projective (X, D) :*

1. *Uniruled \implies weakly uniruled $\implies K_X + D$ not pseudo-effective.*
2. *Rationally connected \implies slope Rationally connected.*

The reverse implications are known only in very special cases when $n > 1$:

3. When (X, D) is Fano, with D reduced (ie: the coefficients of the D'_j s are all equal to 1). Then (X, D) is uniruled ([23]). This means that X is covered by rational curves C meeting D in at most one point (on the normalisation of C). The example of \mathbb{P}^2 with a reduced divisor consisting of 2 lines shows that this is optimal.

4. For $X = \mathbb{P}^n$, with D consisting of $(n + 2)$ hyperplanes with integer multiplicities $m_j, j = 0, \dots, n + 1$, such that (\mathbb{P}^n, D) is Fano, it is shown that (\mathbb{P}^n, D) is uniruled ([11], Theorem 8). The arguments given there have no general character, however. But a counting dimension argument shows that this should ‘heuristically’ hold. We refer to [11], §7 for more similar examples and a more detailed discussion.

12. ORBIFOLD RATIONAL CURVES: A WEAK CONDITIONAL VERSION

Let (X, D) be slope-rationally connected in the sense that there exists a movable classe α such that $\mu_{\pi^*(\alpha), \min}^G(\pi^*(T(X, D))) > 0$ for some (or any) adapted Kawamata-cover $\pi : Y \rightarrow (X, D)$.

The aim of this section is to improve Theorem 1.1 to show that the class α can be chosen to be ‘Geometrically Rational big’ in the sense below, if (X, D) is klt and if the following question has a positive answer:

Question 12.1: *Let (X, D) be a connected and smooth complex-projective orbifold pair such that $K_X + D$ is not pseudo-effective. Does there exist on X an algebraic family of rational curves $(C_t)_{t \in T}$, parametrised by an irreducible projective variety T , whose generic member is irreducible, and such that $-(K_X + D) \cdot C_t > 0$?*

When (X, D) is klt, the known results of the LMMP permit to reduce this question to the following special case:

Question 12.2: *Let (X, D) be a connected and smooth klt complex-projective orbifold pair such that, after a finite number of divisorial contractions and Log-flips $\varphi : (X, D) \rightarrow (X', D')$, with $D' = \varphi_*(D)$, (X', D') is still klt, with X' \mathbb{Q} -factorial, and $K_{X'} + D'$ is Fano with*

²⁰This conjecture is equivalent to Question 9.1 below.

Picard number 1. Does there exist on X an algebraic family of rational curves $(C_t)_{t \in T}$, parametrised by an irreducible projective variety T , whose generic member is irreducible, and such that $-(K_X + D).C_t > 0$?

The problem is to lift a suitable covering family of rational curves on (X', D') to X preserving the negativity of the intersection number with $K + D$. Note that, in general, there is no covering family of rational curves on X' avoiding the non-canonical singularities of (X', D') . It might, however, be true that the canonical singularities of this pair can be avoided.

More might possibly be said in order to characterise slope-Rational connectedness by rational curves (see Question 2 at the end of this section), in analogy with the case when $D = 0$.

The sections §12.1 and §12.2 below are used below only in order to get the ‘bigness’ statement in Theorem 12.9.

12.1. Geometrically Rational classes.

Definition 12.1. *Let X be smooth, complex projective and connected, and $\alpha \in \text{Mov}(X)$. We say that α is ‘Geometrically Rational’ if it belongs to the closed cone $\text{RMov}(X)$ generated by classes of the form $[C]$, for an irreducible rational curve C on X . We say that α is ‘Geometrically Rational big’ if it belongs to the interior $[\text{RMov}(X) \cap \text{Mov}^0(X)]$ of this cone.*

Remark 12.2. *In general, the interior $\text{RMov}^0(X)$ inside $\text{RMov}(X)$ of this cone is contained, but not equal to $[\text{RMov}(X) \cap \text{Mov}^0(X)]$, because, even if X is a rational surface, $\text{RMov}(X)$ is a strict closed subcone (of nonempty interior, see below) of $\text{Mov}(X)$. An exemple is X , given by \mathbb{P}^2 blown-up in 16 points. If C is the strict transform of a generic quintic through these 16 points, then C is easily seen to be ample, but $K_X.C = +1$, which shows that $[C] \notin \text{RMov}(X)$, since $-K_X.\alpha \geq 0$ for any $\alpha \in \text{Mov}(X)$.*

Is it true that $\text{Rmov}(X) = \text{Mov}(X) \cap K_X^{\leq 0}$ if X is rationally connected?

We denote by $K^{<0}(X)$ (resp. $K^{\leq 0}(X)$) be the cone of classes in $N_1(X)$ which have negative (resp. non-negative) intersection with K_X .

Proposition 12.3. *Let X be a connected complex projective manifold. Then:*

1. *$\text{RMov}(X)$ is non-empty if and only if X is uniruled.*
2. *$\text{Mov}(X) \cap K^{<0}(X)$ is non-empty if and only if X is uniruled.*
3. *$\text{RMov}(X) \cap \text{Mov}^0(X)$ is nonempty if and only if X is rationally connected.*
4. *$\text{RMov}(X)$ has nonempty interior in $N_1(X)$ if and only if X is rationally connected.*

Proof. Claim 1 is essentially the definition of uniruledness. Claim 2 is [27].

Claim 3. Assume first that $\alpha \in RMov(X) \cap Mov^0(X)$. There then exists some irreducible rational curve C on X such that $[C] \in RMov^0(X)$, which belongs to an algebraic family of X -covering rational curves $C_t, t \in T$. Assume X were not rationally connected. Let then $r : X \dashrightarrow R$ be its ‘rational quotient’ (which is an almost holomorphic fibration). We then have: $r_*([C]) = 0$. If $H \subset R$ is an effective non-zero divisor, then $H_X := f^{-1}(H)$ does not meet the generic C_t , and $H_X \cdot [C] = 0$, contradicting the fact that $[C] \in Mov^0(X)$. Thus: X is rationally connected if $RMov^0(X) \neq \emptyset$.

In the other direction, assume that X is rationally connected. Let H be an ample divisor on X . There thus exists an integer $d > 0$ such that any two points of X can be joined by an irreducible rational curve C of degree at most d (by the ‘comb deformation technique’ of [25], see [18], Theorem 4.27, p. 105). Let then $[C_s], s = 1, \dots, N$ be the classes of irreducible rational curves of degree at most d belonging to irreducible algebraic families of rational curves connecting two generic two points of X , and let $\alpha := \sum_{s=1}^N [C_s]$. Then $\alpha \cdot D > 0$, for any irreducible effective non-zero divisor $D \subset X$, since $[C_s] \cdot D \geq 0, \forall s$, and $[C_s] \cdot D > 0$ for at least one s (a priori depending on D), by choosing one of the two points outside of D , and C_s irreducible²¹. Thus $\alpha \in RMov^0(X)$, by the following Corollary 12.7 below.

Claim 4. Assume that X is not rationally connected. Claim 3 above then shows that $Rmov(X) \cap Mov^0(X) = \emptyset$. A fortiori, $Rmov(X) \subset Mov(X)$ has no interior point in $Mov^0(X)$.

Conversely, assume that X is rationally connected, and let $\alpha \in Rmov(X) \cap Mov^0(X)$ be as above. It follows again from the ‘comb deformation technique’ of [25] (see [18] for example), that $\alpha + \varepsilon[\Gamma] \in RMov(X)$ for any irreducible rational curve $\Gamma \subset X$ if $\varepsilon > 0$ is sufficiently small. But the K_X -negative part $K^{<0}(X)$ of the closed cone $NE(X)$ of effective curves on X is generated by classes of Mori-extremal rational curves on X , by the ‘cone theorem’ (see [D] for example). Since $K^{<0}(X)$ has nonempty interior in $NE(X)$, any class of the form $\alpha - \varepsilon[\Gamma]$ as above belongs to the interior of $RMov(X)$ in $NE(X)$ if Γ is a Mori-extremal rational curve on X . \square

12.2. Bigness of movable classes. Let X be smooth, connected and complex projective. Let $(C_v)_{v \in V}$ be an algebraic family of curves parametrised by a complex projective irreducible space V . We assume that C_v is irreducible, for $v \in V$ generic, and that the family is X -covering. The class $[C := C_v] \in Mov(X)$ is thus independent of v . The closed cone of $H_2(X, \mathbb{R})$ generated by such classes is $Mov(X)$.

²¹It is certainly possible to choose a single class $[C_s]$ by refining the argument.

Recall that a class $\alpha \in \text{Mov}(X)$ is ‘big’ if it lies in $\text{Mov}^0(X)$, the interior of the cone $\text{Mov}(X)$. This notion is obviously not preserved by blow-ups. We have the obvious:

Lemma 12.4. *$\alpha \in \text{Mov}(X)$ is big if and only if, equivalently:*

1. $\alpha.P > 0$ for any pseudo-effective divisor.
2. $\alpha - \varepsilon.A^{n-1} \in \text{Mov}(X)$ for some $\varepsilon > 0$, if A is some ample divisor on X .

Let $(C_v)_{v \in V}$ be as above, and let $q : \mathbb{P}(TX) \rightarrow X$ be the projectified (by lines) tangent bundle of X . Then each generic C_v has a natural tangential lifting $\widehat{C}_v \subset \mathbb{P}(TX)$. We denote by $[\widehat{C}] \in N_1(\mathbb{P}(TX))$ the corresponding class. then $[\widehat{C}] \in \text{Mov}(\mathbb{P}(TX))$ if, through the generic point $x \in X$ and the tangent direction $\tau \in TX_x$, there exists some C_v going through x with tangent direction τ .

Theorem 12.5. *Assume that $[\widehat{C}] \in \text{Mov}(\mathbb{P}(TX))$. Assume that there exists a non-zero pseudo-effective divisor $P \in N^1(X)$ such that $[C].P = 0$. Then P is \mathbb{Q} -effective.*

The proof will be given in §12.4 below.

Corollary 12.6. *The class $[C_v]$ is big if $[\widehat{C}_v] \in \text{Mov}(\mathbb{P}(TX))$, and either:*

1. $\widehat{C}_v \in \mathbb{P}(TX)$ is ‘strictly movable’ in the sense that any point $z \in \mathbb{P}(TX)$ is contained in some irreducible \widehat{C}_v , or:
2. $[C_v].D > 0$, for any irreducible divisor $D \subset X$.

From this we get immediately:

Corollary 12.7. *Let $[C_s] \in \text{Mov}(X)$, $s = 1, \dots, N$, be such that $[\widehat{C}_s] \in \text{Mov}(\mathbb{P}(TX))$, $\forall s$. If $\alpha := \sum_{s=1}^{s=N} t_s.[C_s]$, $0 \leq t_s \in \mathbb{R}, \forall s$ be such that $\alpha.D > 0$ for any effective non-zero divisor $D \subset X$. Then $\alpha \in \text{Mov}^0(X)$.*

In the case of rational curves, we get:

Corollary 12.8. *Let $[C_s] \in \text{RMov}(X)$, $s = 1, \dots, N$, be such that $[\widehat{C}_s] \in \text{Mov}(\mathbb{P}(TX))$, $\forall s$. Let $\alpha := \sum_{s=1}^{s=N} t_s.[C_s]$, for non-negative real numbers t_s . If $\alpha.D > 0$ for any effective non-zero divisor $D \subset X$, then $\alpha \in \text{RMov}^0(X) \subset (\text{Mov}^0(X) \cap \text{RMov}(X))$.*

Proof. We already know that $\alpha \in (\text{Mov}^0(X) \cap \text{RMov}(X))$. Since $\text{RMov}(X)$ is a subcone of \square

12.3. Slope positivity relative to Rational classes.

We can now improve Theorem 1.1 as follows, in the klt case, but assuming a positive answer to Question 1 above:

Theorem 12.9. *Let (X, D) be a smooth connected klt complex projective orbifold pair which is slope Rationally connected. Assume that the question 1.2 at the beginning of the present section is positive.*

There then exists a ‘Geometrically Rational big’ class α such that $\mu_{\alpha, \min}(\pi^(T(X, D))) > 0$.*

Proof. The proof again works by induction on $n := \dim(X)$. When $n = 1$, the statement is clear, since every big class is then Geometrically Rational. We now assume that the assertion holds true for every $n' < n$. We consider two cases (possibly replacing (X, D) by some of its orbifold birational models:

1. There exists $\alpha \in \text{Mov}^0(X)$ and $\{0\} \subsetneq \mathcal{F} \subsetneq \pi^*(T(X, D))$, saturated, such that $\mu_{\alpha}(\mathcal{F}) > 0$.
2. For every $\alpha \in \text{Mov}^0(X)$ and any saturated $\{0\} \subsetneq \mathcal{F} \subsetneq \pi^*(T(X, D))$, we have: $\mu_{\alpha}(\mathcal{F}) \leq 0$.

Case 1: We then get (after replacing (X, D) by a suitable orbifold birational model), and choosing $\text{rank}(\mathcal{F})$ minimum, a neat fibration which is an orbifold morphism to its orbifold base $f : (X, D) \rightarrow (Z, D_Z)$. And (Z, D_Z) is still sRC, as well as the smooth orbifold fibres (X_z, D_z) of f . We have: $0 < d := \dim(Z) < n$, and thus get, as in the proof of Theorem 1.1, classes $\alpha \in \text{Mov}^0(X/Z)$ and $\beta \in \text{Mov}^0(Z)$ such that $\mu_{\beta, \min}(p^*(T(Z, D_Z))) > 0$, $f_*(\alpha) = 0$, and $\mu_{\alpha, \min}(\pi^*(T(X_z, D_z))) > 0$. We may, by induction on the dimension, assume that $\beta \in \text{RMov}^0(Z)$, and $\alpha_z \in \text{RMov}(X_z)$, where α_z is the restriction to the ‘general’ smooth fibre X_z of f .

The conclusion in this case 1 then follows from (the proof of Theorem 1.1 and) the following:

Lemma 12.10. *Let $f : X \rightarrow Z$ be a fibration between connected complex projective manifolds, together with $\alpha \in \text{Mov}(X/Z)$ and $\beta \in \text{RMov}^0(Z)$ such that $f_*(\alpha) = 0$ and $\alpha_z \in \text{RMov}^0(X_z)$. Then:*

1. *There exists $\beta' \in \text{RMov}(X)$ such that $f_*(\beta') = \beta$, and:*
2. *$\gamma := \varepsilon \cdot \rho + k \cdot \alpha + \beta' \in \text{RMov}^0(X)$ for any $k > 0$ sufficiently large and $\varepsilon > 0$ sufficiently small, if $\rho \in \text{RMov}^0(X)$, which is nonempty by Proposition 12.3, since X is rationally connected.*

Proof. Claim 1: We may reduce to the case where $\beta = [C_t]$ is the class of a Z -covering algebraic family of irreducible rational curves $[C_t], t \in T$. Because the fibres f are rationally connected, it follows from [20] that for t generic, there exists a section C'_t of $f_t : X_t := f^{-1}(C_t) \rightarrow C_t$. Let α_t be the restriction of α to X_t , which makes sense, since $f_*(\alpha) = 0$. From [25], we deduce that $\beta'_t := k_t \cdot \alpha_t + [C'_t]$ is in $\text{RMov}(X_t)$, for $k_t > 0$ sufficiently large, and is thus the class of an X_t -covering family of rational curves of X_t , with $(f_t)_*(\beta'_t) = [C_t]$. From the countability at infinity of the Chow-Barlet space of curves of X , we deduce the existence of a $k > 0$ such that $\beta' := k \cdot \alpha + \beta \in \text{RMov}(X)$ is such that $f_*(\beta') = \beta$, which is Claim 1.

Claim 2: Since $\rho \in RMov^0(X)$ and $\beta' \in RMov(X)$, $\varepsilon.\rho + \beta' \in RMov^0(X)$ for any $\varepsilon > 0$ sufficiently small. On the other hand, we get from the proof of Theorem 1.1 that $\mu_{\pi^*(k.\alpha + \beta'), \min}(\pi^*(T(X, D))) > 0$ for $k > 0$ sufficiently large. It now follows that this remains true for $\varepsilon.\rho + k.\alpha + \beta'$, by Lemma 6.5. \square

Case 2: We seem to need to use [1], here. Let $\psi : X \dashrightarrow X_0$ be a sequence of divisorial contractions and log-flips, with $(X_0, D_0) = (X_0, \psi_*(D))$, such that one has a Log-Fano-contraction $\varphi : (X_0, D_0) \rightarrow Z$ with $(n - d) := \dim(Z) < n$, and of relative Picard number 1. By taking a suitable orbifold birational model of (X, D) , we shall assume that ψ is regular. Choose $\alpha_0 := -(K_{X_0} + D_0)^{d-1}.H^{n-d}$, where $H = \varphi^*(H_Z)$, with H ample on Z . This is a movable curve class on X with $\varphi_*(\alpha_0) = 0$. Let $\psi^*(\alpha_0) := \alpha \in Mov(X)$ be its inverse image. We thus have: $f_*(\alpha) = 0$, if $f = \varphi \circ \psi : X \rightarrow Z$. Let $\mathcal{F} := \pi^*(Ker d(df)) \cap \pi^*(T(X, D))$, if $\pi : Y \rightarrow (X, D)$ is a Kawamata-cover adapted to (X, D) . Then $\mu_\alpha(\mathcal{F}) > 0$, since $f_*(\alpha) = 0$, and $-\alpha.(K_{X_z} + D_z) > 0$, if X_z is a generic fibre of f . We are thus in the first case, unless $\mathcal{F} = \pi^*(T(X, D))$. Since we are in case 2, $\dim(Z) = 0$, that is: (X_0, D_0) is Fano of Picard rank 1, $\pi^*(T(X, D))$ is semi-stable with respect to $\pi^*(\alpha)$, and $\mu_{\pi^*(\alpha), \min}(\pi^*(T(X, D))) > 0$. Question 1.2 having a positive answer, Moreover, (X_0, D_0) being Log-Fano of Picard rank 1, X_0 is covered by an algebraic family of rational curves C'_t of class (proportional to) α_0 such that $-C'_t.(K_{X_0} + D_0) > 0$. Question 1.2 having by assumption a positive answer, there exists a class in $\alpha' \in RMov(X)$ with $-(K_X + D).\alpha' > 0$. The properties shown above for α still hold for α' also, and in particular: $\pi^*(T(X, D))$ is semi-stable with respect to $\pi^*(\alpha')$, with: $\mu_{\pi^*(\alpha'), \min}(\pi^*(T(X, D))) > 0$. Since X is rationally connected, there exists $\rho \in RMov^0(X)$, and $\varepsilon.\rho + \alpha' \in RMov^0(X)$, enjoying these same properties, by the same argument as in case 1, which also concludes the proof in this case. \square

Question 2:

1. Can the big rational class α constructed above be chosen of the form $[C]$, for an irreducible rational curve $C \subset X$ with arbitrary ample normal bundle, and going through any given finite set of points?
2. More importantly, can the rational curve C above be so chosen that $\pi^*(T(X, D))|_{C'}$ is ample, if $C' = \pi^{-1}(C)$? This might be the right definition of a ‘free’ orbifold D -rational curve.

Remark 12.11. *The question 2 might possibly depend on a version of the Grauert-Müllich restriction theorem for the curves of the form C' above and for the vector bundle $\pi^*(T(X, D))$ on them.*

12.4. Proof of the bigness criterion. We prove here the following result, used above:

Theorem 12.12. *Assume that $[\widehat{C}] \in \text{Mov}(\mathbb{P}(TX))$. Assume that there exists a non-zero pseudo-effective divisor $P \in N^1(X)$ such that $[C].P = 0$. Then P is \mathbb{Q} -effective.*

Proof. Let $p : Y \rightarrow X$ be a surjective map with connected fibers between two smooth compact manifolds Y and X of dimension $n+1$ and n , respectively. Given a generic point $y \in Y$, we denote by C_y the fiber $p^{-1}(p(y))$ of p passing through y ; if more than one map p is involved, indices are used in order to distinguish the corresponding fibers.

Proposition 12.13. *(Communicated by M. Păun) Let Y be a $n+1$ -dimensional (smooth) compact complex manifold, and let T be a closed positive $(1,1)$ current on Y . Let surjective maps $p_j : Y \rightarrow X_j$ be given, where X_j is an n -dimensional compact manifold for $j = 1, \dots, n+1$ having the following properties.*

- (i) *There exists a proper analytic set $S \subset Y$ such that for each $y \in Y \setminus S$ the vector space generated by the tangent space of the curves $C_{j,y}$ at y for $j = 1, \dots, n+1$ is $T_{Y,y}$.*
- (ii) *The restriction of the current T to each generic fiber of p_j is equal to zero, for each $j = 1, \dots, n+1$.*

Then we have $\chi_{Y \setminus S} T = 0$.

Proof. Let $S \subset Y$ be an analytic subset of Y , such that the restriction of each p_j to the complement $Y_0 := Y \setminus S$ is a smooth, proper fibration. We show next that we have $\chi_{Y \setminus S} T = 0$.

Let $y \in Y_0$ be an arbitrary point, and let z_1, \dots, z_{n+1} be a set of coordinates at y such that for each $j = 1, \dots, n+1$ the subspace

$$(14) \quad \mathbb{C} \frac{\partial}{\partial z_j}$$

coincides with the tangent space of $C_{j,y}$ at y . The choice of such a coordinate system is possible, due to the hypothesis (i) above.

Locally near y the current T can be written as

$$(15) \quad T|_{\Omega} = \sum_{j,k} T_{j\bar{k}} dz_j \wedge d\bar{z}_k$$

where $T_{j\bar{k}}$ are distributions of order zero on Ω . Let $p : Y \rightarrow X$ be one of the maps above; we recall the following formula of Fubini type (cf. [29])

$$(16) \quad \int_{\Omega} T \wedge p^* \eta = \int_{x \in X} \eta \int_{\Omega \cap p^{-1}(x)} T,$$

where the restriction $T|_{p^{-1}(x)}$ is well defined for almost all $x \in X$, so that the right hand side member in (15) is meaningful. In (17) we denote by η a smooth form of type (n,n) defined (at least) in a open set including $p(\Omega)$.

By the implicit function theorem, there exist $(\eta_j)_{j=1,\dots,n+1}$ a set of smooth (n, n) forms defined in a small open set centered at $p_j(y)$ in X_j such that we have

$$(17) \quad p_j^*(\eta_j) = \xi_1 \wedge \dots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \dots \wedge \xi_{n+1}$$

where we use the notation $\xi_j := \sqrt{-1}dz_j \wedge d\bar{z}_j$.

The Fubini formula (15) combined with the hypothesis (ii) show that

$$(18) \quad \chi_\Omega T \wedge \xi_1 \wedge \dots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \dots \wedge \xi_{n+1} = 0$$

for each $j = 1, \dots, n+1$. In other words, the diagonal distributions $T_{j\bar{j}}$ are identically zero, and so it is the restriction of T to Ω (this is a consequence of the fact that T is positive).

The current T has thus no mass on $Y \setminus S$. \square

Remark 12.14. 1. The hypothesis (ii) above means that $0 = \rho \cdot C_{j,y}, \forall j$, if $\rho := \{T\}$ is the cohomology class of T in $H^{1,1}(X, \mathbb{R})$, since the cohomology class of $T|_{C_{j,y}}$ is the restriction of ρ to $C_{j,y}$.

2. Proposition 12.13 implies that ρ is *effective*, ie: it contains an effective \mathbb{R} -divisor, since $T = \chi_{Y \setminus S} T + \chi_S T$, and so: $T = \chi_S T$. The claim follows from the ‘support theorem’ (see [19]). \square

13. MOTIVATION: THE DECOMPOSITION OF THE CORE MAP.

In [9], Théorème 10.3, we showed the decomposition $c = (J \circ r)^n$ of the ‘core’ map $c : (X, D) \rightarrow (C, D_C)$ of a smooth projective orbifold pair (X, D) . This decomposition was conditional in the ‘ $C_{n,m}^{orb}$ ’-conjecture introduced in [8], §4.1, p. 564. The $C_{n,m}^{orb}$ conjecture was used in order to define the map $r : (X, D) \rightarrow (R^*, D_{R^*})$, its ‘ κ -rational quotient’, for any smooth orbifold (X, D) , while $J : (X, D) \dashrightarrow (J, D_J)$ was a neat model of its ‘Moishezon-Iitaka fibration’ when $\kappa(X, D) \geq 0$. The ‘Slope-Rational Quotient’ $\rho : (X, D) \rightarrow (R, D_R)$ defined above permits to give (unconditionally) a variant of the ‘ κ -rational quotient’. Conjecturally, these two maps actually coincide. We give some details below.

Definition 13.1. ([9], *Définition 5.23*, *Remarque 5.24*)²² Let (X, D) be a smooth (complex projective²³, connected) orbifold pair. Define:

$\kappa^+(X, D) := \max\{\kappa(X', L') | m > 0, L' \subset \otimes^m(\pi^*(\Omega^1(X, D))), rk(L') = 1\}$,
and: $\kappa_+(X, D) := \max\{\kappa(Z, D_Z) | f : (X', D') \rightarrow (Z, D_Z)\}$, where (X', D') is birationally orbifold equivalent to (X, D) , and f is a ‘neat’ orbifold model of (f, D) .

²²The definition (and notation) given there is slightly different, but should lead to the same invariants.

²³Compact complex would suffice for the definitions, here

To simplify notation, we write: $f : (X, D) \dashrightarrow (Z, D_Z)$ for a neat orbifold model of such a fibration.

We obviously have: $\kappa(X, D) \leq \kappa_+(X, D) \leq \kappa^+(X, D)$.

In [9], Corollaire 6.14, we showed, assuming $C_{n,m}^{orb}$, the existence of a unique fibration $r : (X, D) \dashrightarrow (Z, D_Z)$ such that its general orbifold fibres (X_r, D_r) have $\kappa^+(X_r, D_r) = -\infty$, while its orbifold base (R^*, D_{R^*}) had $\kappa(R^*, D_{R^*}) \geq 0$.

We now replace κ by the numerical dimension ν , which usually turns conjectures in theorems.

We showed in [15], that equality holds when we replace $\kappa(X', L')$ by the numerical dimension $\nu(X', L')$ if $K_X + D$ is pseudoeffective: $\nu(X, D) := \nu(X, K_X + D) = \nu^+(X, D)$, the latter being defined as the maximum of $\nu(X', L')$ for the same L' as above.

Since: $\nu^+(X, D) := -\infty$ if and only if: (X, D) is slope-rationally connected, or equivalently, if: $h^0(X', \otimes^m(\pi^*(\Omega^1(X, D)) \otimes A) = 0, \forall k \geq k(A)$, by our main result here, the ‘slope-rational quotient’ $\rho : (X, D) \dashrightarrow (R, D_R)$ defined above unconditionally should coincide with r . The problem one now faces is that $K_R + D_R$ is pseudoeffective, instead of having $\kappa(R^*, D_{R^*}) \geq 0$, as one had with the orbifold base of r . One cannot however define any ‘Moishezon-Iitaka-fibration’ for (R, D_R) without assuming that $\kappa(R, D_R) \geq 0$, if $K_R + D_R$ is only known to be pseudoeffective.

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